Ministry of Basic and Senior Secondary Education

Pupils' Handbook for

Senior Secondary *Mathematics Revision*

PART I

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Please see final page for further information.

Foreword

These Lesson Plans and the accompanying Pupils' Handbooks are essential educational resources for the promotion of quality education in senior secondary schools in Sierra Leone. As Minister of Basic and Senior Secondary Education, I am pleased with the professional competencies demonstrated by the writers of these educational materials in English Language and Mathematics.

The Lesson Plans give teachers the support they need to cover each element of the national curriculum, as well as prepare pupils for the West African Examinations Council's (WAEC) examinations. The practice activities in the Pupils' Handbooks are designed to support self-study by pupils, and to give them additional opportunities to learn independently. In total, we have produced 516 lesson plans and 516 practice activities – one for each lesson, in each term, in each year, for each class. The production of these materials in a matter of months is a remarkable achievement.

These plans have been written by experienced Sierra Leoneans together with international educators. They have been reviewed by officials of my Ministry to ensure that they meet the specific needs of the Sierra Leonean population. They provide step-by-step guidance for each learning outcome, using a range of recognized techniques to deliver the best teaching.

I call on all teachers and heads of schools across the country to make the best use of these materials. We are supporting our teachers through a detailed training programme designed specifically for these new lesson plans. It is really important that the Lesson Plans and Pupils' Handbooks are used, together with any other materials they may have.

This is just the start of educational transformation in Sierra Leone as pronounced by His Excellency, the President of the Republic of Sierra Leone, Brigadier Rtd Julius Maada Bio. I am committed to continue to strive for the changes that will make our country stronger and better.

I do thank our partners for their continued support. Finally, I also thank the teachers of our country for their hard work in securing our future

Mr. Alpha Osman Timbo

Minister of Basic and Senior Secondary Education

The policy of the Ministry of Basic and Senior Secondary Education, Sierra Leone, on textbooks stipulates that every printed book should have a lifespan of three years.

To achieve thus, DO NOT WRITE IN THE BOOKS.

Table of Contents

Lesson 1: Basic Numeration 5

Lesson 2: Sequences 9

Lesson 3: Series 13

Lesson 4: Problem Solving Using Sequences and Series 17

Lesson 5: Ratios 20

Lesson 6: Rates 24

Lesson 7: Proportional Division 28

Lesson 8: Speed 31

Lesson 9: Application of Percentages – Part 1 35

Lesson 10: Application of Percentages – Part 2 39

Lesson 11: Application of Percentages – Part 3 43

Lesson 12: Indices 47

Lesson 13: Indices 51

Lesson 14: Logarithms 56

Lesson 15: Representing Sets with Diagrams and Symbols 60

Lesson 16: Solving Problems Involving Sets 64

Lesson 17: Operations on Surds 68

Lesson 18: Simplifying Surds 72

Lesson 19: Simplification and Factorisation 76

Lesson 20: Functions 80

Lesson 21: Graphing Linear Functions 84

Lesson 22: Applications of Linear Functions 88

Lesson 23: Distance and Mid-point Formulae 92

Lesson 24: Graphing and Interpreting Quadratic Functions 95

Lesson 25: Solving Quadratic Equations Algebraically – Part 1 99

Lesson 26: Solving Quadratic Equations Algebraically – Part 2 103

Lesson 27: Problem Solving with Quadratic Equations 107

Lesson 28: Simultaneous Linear Equations 110

Lesson 29: Applications of Simultaneous Linear Equations 114

Lesson 30: Simultaneous Quadratic and Linear Equations 118

Lesson 31: Tangent to a Quadratic Function 122

Lesson 32: Inequalities 126

Lesson 33: Variation 129

Lesson 34: Simplification of Algebraic Fractions 133

Lesson 35: Operations on Algebraic Fractions 135

Lesson 36: Logical Reasoning – Part 1 139

Lesson 37: Logical Reasoning – Part 2 143

Lesson 38: Pie Charts and Bar Charts 147

Lesson 39: Mean, Median, and Mode of Ungrouped Data 151

Lesson 40: Histograms 155

Lesson 41: Frequency Polygons 159

Lesson 42: Mean, Median, and Mode of Grouped Data 163

Lesson 43: Cumulative Frequency Curves and Quartiles 167

Lesson 44: Percentiles 171

Lesson 45: Measures of Dispersion 175

Lesson 46: Standard Deviation 178

Lesson 47: Mean Deviation 182

Lesson 48: Statistics Problem Solving 186

Answer Key: Term 1 190

Appendix I: Protractor 216

Introduction

to the Pupils' Handbook

These practice activities are aligned to the Lesson Plans, and are based on the National Curriculum and the West Africa Examination Council syllabus guidelines. They meet the requirements established by the Ministry of Basic and Senior Secondary Education.

1. The practice activities will not take the whole term, so use any extra time to revise material or re-do activities where you made mistakes.
2. Use other textbooks or resources to help you learn better and practise what you have learned in the lessons.
3. Read the questions carefully before answering them. After completing the practice activities, check your answers using the answer key at the end of the book.
4. Make sure you understand the learning outcomes for the practice activities and check to see that you have achieved them. Each lesson plan shows these using the symbol to the right.

Learning Outcomes

5. Organise yourself so that you have enough time to complete all of the practice activities. If there is time, quickly revise what you learned in the lesson before starting the practice activities. If it is taking you too long to complete the activities, you may need more practice on that particular topic.
6. Seek help from your teacher or your peers if you are having trouble completing the practice activities independently.
7. Make sure you write the answers in your exercise book in a clear and systematic way so that your teacher can check your work and you can refer back to it when you prepare for examinations.
8. Congratulate yourself when you get questions right! Do not worry if you do not get the right answer – ask for help and continue practising!

KEY TAKEAWAYS FROM SIERRA LEONE'S PERFORMANCE IN WEST AFRICAN SENIOR SCHOOL CERTIFICATE EXAMINATION – GENERAL MATHEMATICS[1]

This section, seeks to outline key takeaways from assessing Sierra Leonean pupils' responses on the West African Senior School Certificate Examination. The common errors pupils make are highlighted below with the intention of giving teachers an insight into areas to focus on, to improve pupil performance on the examination. Suggestions are provided for addressing these issues.

Common errors

1. Errors in applying principles of BODMAS
2. Mistakes in simplifying fractions
3. Errors in application of Maths learned in class to real-life situations, and vis-a-versa.
4. Errors in solving geometric constructions.
5. Mistakes in solving problems on circle theorems.
6. Proofs are often left out from solutions, derivations are often missing from quadratic equations.

Suggested solutions

1. Practice answering questions to the detail requested
2. Practice re-reading questions to make sure all the components are answered.
3. If possible, procure as many geometry sets to practice geometry construction.
4. Check that depth and level of the lesson taught is appropriate for the grade level.

[1] This information is derived from an evaluation of WAEC Examiner Reports, as well as input from WAEC examiners and Sierra Leonean teachers.

It is important that candidates understand what to expect on the day of the WASSCE exam. Details of the exam and strategies for taking the exam are given below.

Content of the WASSCE Exam

The WASSCE Mathematics exam consists of 3 sections. These are described in detail below:

Paper 1 – Multiple Choice

- Paper 1 is 1.5 hours, and consists of 50 multiple choice questions. It is worth 50 marks.
- This gives 1.8 minutes per problem, so time must be planned accordingly.
- The questions are drawn from all topics on the WASSCE syllabus.

Paper 2 – Essay Questions

- Paper 2 consists of 13 essay questions in 2 sections – 2A and 2B.
- Paper 2 is worth 100 marks in total.
- Pupils will be required to answer 10 essay questions in all, across the 2 sections.
- This is an average of 15 minutes per question. However, keep in mind that section 2B is more complicated, so plan your time accordingly.

Paper 2A – Compulsory Questions

- Paper 2A is worth 40 marks.
- There are 5 **compulsory** essay questions in paper 2A.
- Compulsory questions often have multiple parts (a, b, c, …). The questions may not be related to each other. Each part of the question should be completed.
- The questions in paper 2A are simpler than those in 2B, generally requiring fewer steps.
- The questions in paper 2A are drawn from the common area of the WASSCE syllabus (they are not on topics that are specific to certain countries).

Paper 2B – Advanced Questions

- Paper 2B is worth 60 marks.
- There are 8 essay questions in paper 2A, and candidates are expected to answer 5 of them.
- Questions in section 2B are of a greater length and difficulty than section 2A.
- A maximum of 2 questions (from among the 8) may be drawn from parts of the WASSCE syllabus that are not meant for Sierra Leone. These topics are not in the Sierra Leone national curriculum for secondary schools. Candidates

from Sierra Leone may choose to answer such questions, but it is not required.
- Choose 5 questions on topics that you are more comfortable with.

Exam Day

- Candidates should bring a pencil, geometry set, and scientific calculator to the WASSCE exam.
- Candidates are allowed to use log books (logarithm and trigonometry tables), which are provided in the exam room.

Exam-taking skills and strategies

- Candidates should read and follow the instructions carefully. For example, it may be stated that a trigonometry table should be used. In this case, it is important that a table is used and not a calculator.
- Plan your time. Do not spend too much time on one problem.
- For essay questions, show all of your working on the exam paper. Examiners can give some credit for rough working. Do not cross out working.
- If you complete the exam, take the time to check your solutions. If you notice an incorrect answer, double check it before changing it.
- For section 2B, it is a good idea to skim read the questions quickly before getting started. If you see a topic that you are familiar and comfortable with, work on those problems. Try not to spend a lot of time deciding which problems to solve, or thinking about problems you will not solve.

Lesson Title: Basic numeration	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L001	**Class:** SSS 4

Learning Outcomes

By the end of the lesson, you will be able to:

1. Apply the principles of BODMAS to operations on rational numbers.
2. Approximate answers to a given number of decimal places and significant figures.
3. Calculate the percentage error using rounded values.

Overview

BODMAS

The letters BODMAS stand for: Bracket, Of (or 'order'), Division, Multiplication, Addition and Subtraction. When working problems which have more than one operation (of, ×, +, –, ÷), we use BODMAS. The letters of the word (BODMAS) tell us the order in which we should work the operations in a Maths problem. The term 'of" represents the multiplication sign, which includes powers.

Approximation

You may be asked to give a result to a certain number of decimal places or significant figures. The rules for approximation of these are nearly the same. If the digit after the place you are rounding to is less than 5, just discard the latter digits (or change them to zeros in the case of significant figures). If this digit is 5 or more, add 1 to the digit in the next place.

You may be asked to round decimals to a stated number of decimal places, or to the tenths, hundredths, or thousandths place. The 1st decimal place is the tenths digit, the 2nd is the hundredths, the 3rd is the thousandths, and so on.

Recall the rules for identifying significant figures (s.f.):

Rules	Examples	Number of s.f.
1. All non-zero digits are significant.	123	(3 s.f.)
2. Zeros between non-zero digits are significant	12.507 304	(5 s.f.) (3 s.f.)
3. Zeros to the left of the first non-zero digit are not significant.	1.02 0.12 0.012	(3 s.f.) (2 s.f.) (2 s.f.)
4. If a number ends in zeros to the right of the decimal point, those zeros are significant.	2.0 2.00	(2 s.f.) (3 s.f.)
5. Zeros at the end of a whole number are not significant, unless there is a decimal point.	4300 4300.0	(2 s.f.) (5 s.f.)

Percentage Error

When we make an estimation, percentage error tells us how close to the exact value our estimated value is. A smaller percentage error means our estimate is **more** accurate, and a larger percentage error means our estimate is **less** accurate.

When using measurements, estimating with smaller units of measure gives us a smaller percentage error. For example, estimating with centimetres is more accurate than estimating with metres. The percentage error will be smaller.

The percentage error is calculated from another amount called simply "error". We calculate the error first, then the percentage error.

How to find an error:

1. Find the range of numbers that a rounded quantity could fall between.
   Example: If an estimated measurement is 2.5 metres to 2 s.f., then the actual measurement could lie in the range 2.45 m to 2.55 m.
2. Error is the difference between the estimated amount, and the minimum and maximum possible amounts that the actual quantity could be.
   Example: In the case above, the error is $2.55 - 2.5 = +0.05\text{ m}$ or $2.45 - 2.5 = -0.05\text{ m}$. This is written as $\text{error} = \pm 0.05\text{ m}$.
3. Error is usually written with a $\pm$ symbol.

In some problems, you will be given an actual value and an estimated or measured value (see solved example 5). In such cases, the error can be found by simply subtracting the actual value from the estimated or measured value.

How to find the percentage error:

The percentage error is found by finding the error as a percentage of the actual measurement. The formula is:

$$\text{Percentage error} = \frac{\text{error}}{\text{actual value}} \times 100\%$$

In cases where you don't have the actual value (see solved example 4), write the measured value in the denominator.

Solved Examples

1. Evaluate $3.6 + 6 \times (5 + 4) \div 3 - 7.15$. Give your answer correct to 1 decimal place.
   Solution:

$3.6 + 6 \times (5 + 4) \div 3 - 7$	$= 3.6 + 6 \times 9 \div 3 - 7.15$	Bracket
	$= 3.6 + 54 \div 3 - 7.15$	Multiply
	$= 3.6 + 18 - 7.15$	Divide
	$= 21.6 - 7.15$	Add
	$= 14.45$	Subtract
	$= 14.5$	Round

2. Evaluate $\left(2\frac{1}{3}+4\frac{1}{2}\right)\div 8\frac{1}{5}$

   Solution:

   Perform the operation in the bracket first:

   $$2\frac{1}{3}+4\frac{1}{2}=\frac{7}{3}+\frac{9}{2}=\frac{14+27}{6}=\frac{41}{6}$$

   We now have $\frac{41}{6}\div 8\frac{1}{5}$:

   $$\frac{41}{6}\div 8\frac{1}{5}=\frac{41}{6}\div\frac{41}{5}=\frac{41}{6}\times\frac{5}{41}=\frac{5}{6}$$

3. Approximate the following to 3 significant figures:
   a. 0.00538970
   b. 34,506
   c. 3.001
   d. 245.67

   Solutions:
   a. 0.00539
   b. 34,500
   c. 3.00
   d. 246

4. The width of a room is measured, and is found to be 2.5 meters to 2 s.f. What is the percentage error?

   Solution:

   If the width has been rounded to 2.5, then the range of values that the width could actually fall into is 2.45 – 2.55. Therefore:

   $$\text{error} = 2.55 - 2.5 = +0.05\text{ m}$$
   or $$\text{error} = 2.45 - 2.5 = -0.05\text{ m}$$

   $$\text{Percentage error} = \frac{\text{error}}{\text{measurement}}\times 100\% = \frac{\pm 0.05\text{ m}}{2.5\text{ m}}\times 100\% = \pm 2\%$$

5. A carpenter measured a plank which is 1.44 m long, to be 1.56 m. What is the percentage error?

   Solution:

   Error = [illegible]

   Percentage error = $\frac{\text{Error}}{\text{Actual value}}\times 100\% = \frac{0.12}{1.44}\times 100\% = 8.33\%$

Practice

1. Simplify: $10\frac{2}{5} - 6\frac{2}{3} + 3$ without using a calculator or tables.
2. Without using calculators or tables, simplify: $37\frac{1}{2} \div \frac{5}{9}\ of \left(\frac{4}{7} + \frac{1}{5}\right) - 80\frac{1}{3}$
3. Write 14.085 correct to two decimal places.
4. Find the value of 25.3×1.5 correct to 1 decimal place.
5. Correct 0.005361 to 2 significant figures.
6. Find the value of $6.25 \div 1.5$ correct to 4 significant figures.
7. A stick is $20\ cm$ long. A girl measured it to be $20.25cm$ long. Find the percentage error of her measurement.
8. A boy estimated his transport fare for a journey as Le $198{,}000.00$ instead of the correct fare, which is Le $200{,}000.00$. Find the percentage error in his estimate.
9. Find the percentage error in approximating $\frac{1}{8}$ as 0.13.
10. A carpenter measured the length of a stick to be $35\ cm$ making an error of 12.5%. What is the actual length of the rod, if it is more than that measured by the carpenter?

Lesson Title: Sequences	Theme: Numbers and Numeration
Practice Activity: PHM4-L002	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Identify arithmetic and geometric sequences.
2. Apply the formulae to find the nth term of a sequence.

Overview

Arithmetic Progressions

A sequence is a list of numbers that follows a certain rule. A sequence in which the terms either increase or decrease by a common difference is an arithmetic sequence, or arithmetic progression. It can be abbreviated as AP.

Consider the sequence: 5, 7, 9, 11, 13, …

The first term is 5 and the **common difference** is 2. The common difference is a difference that is the same between each term and the next term.

The letter a is commonly used to describe the first term, and the letter d is used for common difference. The general arithmetic progression is given as:

$$a, a+d, a+2d, a+3d, \ldots$$

The first term is a, and a difference of d is added to each subsequent term.

The below formula gives the general term of an AP. That is, it describes every term in the sequence. It can be used to find the nth term of an AP:

$U_n = a + (n-1)d$, where U_n is the nth term of the AP.

Geometric Progressions

A sequence in which the terms either increase or decrease by a common ratio is a geometric sequence, or geometric progression. It can be abbreviated to GP.

Consider the sequence: 1, 2, 4, 8, 16, 32, …

The first term is 1 and the **common ratio** is 2. The common ratio is multiplied by each term to get the next term.

The letter a is commonly used to describe the first term, and the letter r is used for the common ratio. The general geometric progression is given as:

$$a, ar, ar^2, ar^3, \ldots$$

where the first term is a, and the ratio r is multiplied by each subsequent term.

Note these special cases for GPs:

- If the numbers decrease as the GP progresses, the common ratio must be a fraction. (Example: The sequence $32, 16, 8, 4, ...$, where $r = \frac{1}{2}$.)
- If the numbers alternate between positive and negative digits, the common ratio must be a negative value. (Example: The sequence $1, -2, 4, -8, 16, ...$, where $r = -2$)

The below formula gives the general term of a GP. That is, it describes every term in the sequence. It can be used to find the nth term of a GP:

$U_n = ar^{n-1}$, where U_n is the nth term of the GP.

Solved Examples

1. Find the 9th term of the sequence $5, 9, 13, 17, ...$

Solution:

The first term is $a = 5$, the common difference is $d = 4$. This is an arithmetic progression, therefore apply the formula for AP:

$$\begin{aligned} U_n &= a + (n-1)d && \\ U_9 &= 5 + (9-1)4 && \text{Substitute for } a, d \text{ and } n \\ &= 5 + (8)(4) && \text{Clear the brackets} \\ &= 37 && \text{Simplify} \end{aligned}$$

2. Find the 7th term of the sequence. $24, 19, 14, 9, ...$

Solution:

The first term is $a = 24$, the common difference is $d = -5$. This is an arithmetic progression, therefore apply the formula for AP:

$$\begin{aligned} U_n &= a + (n-1)d && \\ U_7 &= 24 + (7-1)(-5) && \text{Substitute for } a, d \text{ and } n \\ &= 24 + (6)(-5) && \\ &= -6 && \text{Simplify} \end{aligned}$$

3. The fifth term of a sequence is 27, and the common difference is 5. Find the eighth term.

Solution:

We know this is an arithmetic sequence, because it has a common difference. To find the eighth term of the sequence, we need to know the value of a, the first term. We can find this with the information given.

Step 1. Find a:

Given $U_5 = 27$ and $d = 5$,

$$\begin{aligned} U_n &= a + (n-1)d && \\ 27 &= a + (5-1)(5) && \text{Substitute } U_5 = 27, n = 5, d = 5 \\ 27 &= a + 20 && \text{Clear the brackets} \\ a &= 7 && \text{Solve for } a \end{aligned}$$

Step 2. Find the eighth term:

$$
\begin{aligned}
U_8 &= 7+(8-1)5 && \text{Substitute } a=7, d=5, n=8 \\
U_8 &= 42 && \text{Clear the brackets and simplify}
\end{aligned}
$$

The eighth term is 42.

4. Find the sixth term of the sequence $1, 2, 4, 8, \ldots$

   Solution:

   This is a geometric progression with the common ratio $r = 2$. To find the sixth term of the sequence, write down the formula for the n^{th} term of a GP and substitute the given values.

$$
\begin{aligned}
U_n &= ar^{n-1} \\
U_6 &= 1 \times 2^{6-1} && \text{Substitute } a=1, n=6, r=2 \\
&= 1 \times 2^5 && \text{Simplify} \\
&= 32
\end{aligned}
$$

The sixth term is 32.

5. Find the eighth term of the sequence $6, -3, \frac{3}{2}, -\frac{3}{4}, \ldots$

   Solution:

   This is a geometric progression with the common ratio $r = -\frac{1}{2}$. To find the eighth term of the sequence, write down the formula for the n^{th} term of a GP and substitute the given values.

$$
\begin{aligned}
U_n &= ar^{n-1} \\
U_8 &= 6 \times \left(-\frac{1}{2}\right)^{8-1} && \text{Substitute } n=8, r=-\frac{1}{2}, a=6 \\
&= 6 \times \left(-\frac{1}{2^7}\right) && \text{Simplify} \\
&= -\frac{6}{128} \\
&= -\frac{3}{64}
\end{aligned}
$$

The eighth term is $-\frac{3}{64}$.

6. The seventh term of a geometric progression is 192. If the first term is 3, find the fifth term.

   Solution:

   To find the fifth term of the sequence, we need to know the value of r, the common ratio. We can find this with the information given.

   Step 1. Find the value of r:

$$
\begin{aligned}
U_n &= ar^{n-1} \\
192 &= 3 \times r^{7-1} && \text{Substitute } U_7 = 192, a=5, n=7 \\
64 &= r^6 && \text{Divide throughout by 3 and simplify} \\
2^6 &= r^6 \\
r &= 2 && \text{Solve for } r
\end{aligned}
$$

Step 2. Find the fifth term:

$$
\begin{aligned}
u_5 &= 3 \times 2^{5-1} && \text{Substitute } a = 3, r = 6, n = 5 \\
&= 48
\end{aligned}
$$

The fifth term is 48.

7. Find the number of terms in the arithmetic progression $5, 8, 11, \ldots, 62$

   Solution:

   To find the number of terms in the AP, write down the formula for the n^{th} term of an AP and substitute the given values.

$$
\begin{aligned}
U_n &= a + (n-1)d \\
62 &= 5 + (n-1)3 && \text{Substitute } U_n = 62, a = 5, d = 3 \\
62 - 5 &= (n-1)3 && \text{Transpose } 5. \\
n &= 20 && \text{Solve for } n
\end{aligned}
$$

There are 20 terms in the progression.

8. The nth term of a sequence is $T_n = 3 + (n-1)^2$. Evaluate $T_5 - T_3$.

   Solution:

   This is neither an arithmetic nor geometric sequence. It does not have a common ratio or difference. Such problems appear on the WASSCE exam. Generally, a formula is given, and you must apply problem-solving skills.

   Step 1. Find T_5 and T_3:

$$
\begin{aligned}
T_5 &= 3 + (5-1)^2 && \text{Substitute } n = 5 \\
T_5 &= 3 + 4^2 && \text{Simplify} \\
T_5 &= 3 + 16 = 19 \\
T_3 &= 3 + (3-1)^2 && \text{Substitute } n = 3 \\
T_3 &= 3 + 2^2 && \text{Simplify} \\
T_3 &= 3 + 4 = 7
\end{aligned}
$$

   Step 2. Subtract: $T_5 - T_3 = 19 - 7 = 12$

Practice

1. Find the sixth term of the sequence 15, 21, 27, 33, …
2. Find the seventh term of the arithmetic sequence 3, -2, -7, -12, …
3. Find the number of terms in the arithmetic sequence 7,11,15, … , 79.
4. A nth term of a sequence is given as $U_n = 1 + 2U_{n-1,}$ for $n \geq 1$. If $U_1 = 8$, find U_3.
5. The 6^{th} term of an arithmetic progression is 35 and the 13^{th} term is 77. Find the 20^{th} term.
6. Find the 8^{th} term of the geometric sequence, $3, 9, 27, 51, \ldots$
7. Write down the 9^{th} term of the GP $3, -2, \frac{4}{3}, \ldots$
8. Find the number of terms in the geometric sequence 2,4,8, … , 512.
9. Find the n^{th} term of the sequence 5,10, 20, 40, ...

Lesson Title: Series	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L003	**Class:** SSS 4

Learning Outcomes

By the end of the lesson, you will be able to:

1. Distinguish between sequence and series
2. Calculate the sum of the first n terms of an arithmetic and a geometric series

Overview

When the terms of a sequence are added together, the result is a series.

Some series carry on forever. These are called infinite series. It is often impossible to find the sum of infinite series, but we can find the sum of the first n terms. Some series have a certain number of terms. These are called finite series. They do not carry on forever, but end at a certain point. It is always possible to find the sum of a finite series.

The following formula is used to find the sum of the first n terms of an AP:

$$S_n = \frac{1}{2}n[2a + (n - 1)d]$$

Remember that the variable n gives the number of terms, a gives the first term, and d gives the common difference.

There are 2 formulae for finding the sum of a GP. The following formula is used to find the sum of the first n terms of a GP when r is a fraction $|r| < 1$:

$$S_n = \frac{a(1-r^n)}{1-r}$$

The following formula is used to find the sum when $|r| > 1$:

$$S_n = \frac{a(r^n-1)}{r-1}$$

Remember that the variable n gives the number of terms, a gives the first term, and r gives the common ratio.

Solved Examples

1. Find the sum of the first 16 terms of the series: $3 + 7 + 11 + 15 + \cdots$
   Solution:
   The series is an AP with a common difference of 2.
   Given: $a = 3, d = 4, n = 16$

S_n	$=$	$\frac{1}{2}n[2a + (n - 1)d]$	
	$=$	$\frac{1}{2}(16)[2(3) + (16 - 1)(4)]$	Substitute n, a, and d
	$=$	$(8)[6 + 60]$	Simplify
	$=$	528	

The sum of the first sixteen terms is 528.

2. Find the sum of the first nine terms of the AP: $12 + 11\frac{1}{2} + 11 + 10\frac{1}{2} + \cdots$

Solution:

Given: $a = 12, d = -\frac{1}{2}, n = 9$

$$S_n = \frac{1}{2}n[2a + (n-1)d]$$

$$S_9 = \frac{1}{2}(9)[2(12) + (9-1)(-\tfrac{1}{2})] \quad \text{Substitute } n, a, \text{ and } d$$

$$= \left(\frac{9}{2}\right)\left[24 - \frac{8}{2}\right] \quad \text{Simplify}$$

$$= \left(\frac{9}{2}\right)[24 - 4]$$

$$= \left(\frac{9}{2}\right)(20)$$

$$= 90$$

The sum of the first nine terms is 90.

3. Find the sum of the first eight terms of the series $4 + 12 + 36 + 108 + \cdots$

Solution:

The series is a GP with common ratio of 3.

Given: $a = 4, r = 3, n = 8$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_8 = \frac{4(3^8 - 1)}{3 - 1} \quad \text{Substitute } n, a, \text{ and } r$$

$$= \frac{4}{2}(3^8 - 1)$$

$$= (2)(6{,}561 - 1) \quad \text{Simplify}$$

$$= 13{,}120$$

The sum of the first eight terms is $13{,}120$.

4. Find the sum of the first 6 terms of the series: $24, -12, +6, -3, + \cdots$

Solution:

The series is a GP with common ratio of $-\frac{1}{2}$.

Given: $a = 24, r = -\frac{1}{2}, n = 6$

$$S_n = \frac{a(1 - r^n)}{1 - r} \quad \text{Since } |r| < 1$$

$$S_6 = \frac{24\left(1 - \left(-\frac{1}{2}\right)^6\right)}{1 - (-\frac{1}{2})} \quad \text{Substitute } n, a, \text{ and } r$$

$$= \frac{24\left(1 - \frac{1}{64}\right)}{1 + \frac{1}{2}} \quad \text{Simplify}$$

$$= \frac{24\left(\frac{64}{64} - \frac{1}{64}\right)}{1 + \frac{1}{2}}$$

$$= \frac{24\left(\frac{63}{64}\right)}{\frac{3}{2}}$$

$$= \frac{3\left(\frac{63}{8}\right)}{\frac{3}{2}}$$

$$\frac{189}{8} \div \frac{3}{2} = \frac{189}{8} \times \frac{2}{3}$$

$$= 15\frac{3}{4}$$

The sum of the first 6 terms is $15\frac{3}{4}$.

5. The sum of the first six terms of a GP is 315. If the common ratio is 2, find the third term.

Solution:

Recall the formula from the previous lesson, $U_n = ar^{n-1}$. To find the third term, you need to know the first term a, and the common ratio r. The common ratio is given but the first term is not given. Use what is given in the question, (sum and number of terms), to find the first term.

Given: $S_6 = 315, r = 2, n = 6$

S_n	$=$	$\frac{a(r^n-1)}{r-1}$	Since $\|r\| > 1$
315	$=$	$\frac{a(2^6-1)}{2-1}$	Substitute s_6,n, and r
	$=$	$\frac{a(64-1)}{1}$	Simplify
315	$=$	$a(63)$	
a	$=$	5	Solve for a

To find the third term, use the formula for the n^{th} term of a GP.

$$U_n = ar^{n-1}$$

$$U_3 = 5(2)^{3-1}$$

$$= 20$$

The third term is 20.

6. The first term of a GP is 5 and the common ratio is 3. Find the number of terms that gives a sum of 605.

Solution:

To find the number of terms, use the formula for the sum of the first n terms of a GP.

Given: $S_n = 605, r = 3, a = 5$

S_n	$=$	$\frac{a(r^n-1)}{r-1}$	Since $\|r\| > 1$
605	$=$	$\frac{5(3^n-1)}{3-1}$	Substitute s_n,a and r
605	$=$	$\frac{5(3^n-1)}{2}$	Simplify
$1{,}210$	$=$	$5(3^n - 1)$	Multiply throughout by 2
242	$=$	$3^n - 1$	Divide throughout by 5
3^n	$=$	243	Solve for n

$$n = 5$$

To number of terms that sum up to 605 is 5.

7. The sum of the first six terms of an AP is 225, if the common difference is 5:
   a. Find the first term.
   b. Write the first three terms of the AP.

   Solution:

   a.

   Given: $S_6 = 225, d = 5, n = 6$

   $$S_n = \frac{n}{2}[2a + (n-1)d]$$

   $$225 = \frac{6}{2}[2a + (6-1)5]$$ Substitute s_6, d and n.

   $$75 = 2a + 25$$ Divide throughout by 3 and simplify

   $$50 = 2a$$

   $$a = 25$$ Solve for a.

   The first term is 25.

   b. We can simply add the common difference to find the next term. The first three terms are $25, 30, 35$.

Practice

1. Find the sum of the first seven terms of the arithmetic series $8 + 12 + 16 +$......
2. Find the sum of the first eight terms of the series $25 + 17 + 9 + \ldots\ldots\ldots$
3. Find the sum of the first ten terms of the AP $3\frac{3}{4} + 6\frac{1}{4} + 8\frac{3}{4} + \ldots\ldots\ldots$
4. Find the sum of the first five terms of the geometric series $1 + 3 + 9 + \ldots\ldots\ldots$
5. Find the sum of the first ten terms of the series $6 - 12 + 24 - 48 + \ldots\ldots\ldots$..
6. Find the sum of the first seven terms of the sequence $24, -12, 6, -3, \ldots\ldots\ldots$
7. The sum of the first three terms of an AP is 24 and the sum of the first seven terms is 98. Find the:
   a. First term;
   b. Common difference;
   c. Sum of the first nine terms of the AP.

Lesson Title: Problem solving using sequences and series	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L004	**Class:** SSS 4

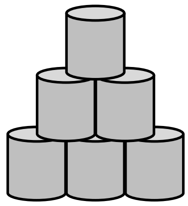
Learning Outcome
By the end of the lesson, you will be able to apply sequences and series to numerical and real-life problems.

Overview

This lesson applies the content on sequence and series from previous lessons to solve real-life problems.

Solved Examples

1. Mr. Bangura sells cans of fish in the market. He decided to display them in a nice way to attract more customers. He arranged them in a stack so that 1 can is in the top row, 2 cans are in the next row, 3 cans are in the third row, and so on. The bottom row has 16 cans of fish. The top 3 rows are shown below.
   a. Write a sequence based on this story.
   b. What is the total number of cans in the stack?



   Solutions:
   a. 1, 2, 3, 4, 5, …, 16.
   b. Calculate the sum of the series using $a = 1$, the number of cans in the first row; $n = 16$, the number of rows; and $d = 1$, the difference between each row and the next row.

S_n	$=$	$\frac{1}{2}n[2a + (n-1)d]$	
S_{16}	$=$	$\frac{1}{2}(16)[2(1) + (16-1)1]$	Substitute n, a, and d
	$=$	$8(2 + 15)$	Simplify
	$=$	$8(17) = 136$	

   Answer: There are 136 cans in the stack.

2. Adama is employed as a secretary with a starting monthly salary of Le $450{,}000.00$. She will receive an increase of Le $15{,}000.00$ at the end of every month. How many months will she work before receiving a monthly salary of Le $540{,}000.00$?
   Solution:
   The first payment is Le $450{,}000.00$ with an increase of Le $15{,}000.00$ every month. Thus, her monthly salary will be Le $450{,}000.00$, Le $(450{,}000.00 + 15{,}000.00)$, Le $(450{,}000.00 + 15{,}000.00 + 15{,}000.00)$, and so on.

   This is an arithmetic progression where the first term is Le $450{,}000.00$, and the common difference is Le $15{,}000.00$.

Given: $a = 450{,}000, d = 15{,}000$, and $U_n = 540{,}000$

$$
\begin{aligned}
U_n &= a + (n-1)d \\
540{,}000 &= 450{,}000 + (n-1)15{,}000 && \text{Substitute } a\text{,d, and } U_n \\
540{,}000 - 450{,}000 &= (n-1)15{,}000 && \text{Transpose } 450{,}000 \\
90{,}000 &= 15{,}000n - 15{,}000 && \text{Solve for } n \\
105{,}000 &= 15{,}000n \\
n &= 7
\end{aligned}
$$

Adama will work for 7 months before receiving a monthly salary of Le $540{,}000.00$.

3. A room has eight boxes. In the first box, there are 2 bags of rice, in the second box there are 6 bags, in the third box 18 bags and so on. How many bags of rice are in the room?

Solution:

There are eight boxes in the room. The number of bags of rice in each box are as follows:

1^{st} box = 2 bags, 2^{nd} box = 6 bags, 3^{rd} box= 18 bags

This gives the geometric series: $2 + 6 + 18 + \ldots$ where $a = 2$, $r = \frac{6}{2} = \frac{18}{6} = 3$, and $n = 8$.

$$
\begin{aligned}
S_n &= \frac{a(r^n-1)}{r-1} && \text{Since } r > 1 \\
S_8 &= \frac{2(3^8-1)}{3-1} && \text{Substitute } a, n, \text{ and } r \\
&= (6{,}561 - 1) && \text{Simplify} \\
&= 6{,}560
\end{aligned}
$$

Therefore, there are $6{,}560$ bags of rice in the room.

4. A trader borrows \$$1{,}600.00$ and has to repay with a total interest of \$$320.00$. If his first monthly repayment is \$$50.00$ and each payment is more than the preceding one by \$$20.00$, how many months will he take to repay the loan and interest?

Solution:

The amount the trader has to repay is the loan and the interest which is $\$1{,}600.00 + \$320.00 = \$1{,}920.00$.

The condition for the payment is that he pays \$$50.00$ for the first payment, the second payment is \$$(50.00 + 20.00)$, the third payment is \$$(50.00 + 20.00 + 20.00)$, and so on.

This gives the arithmetic series: $50 + 70 + 90 + \cdots$ where $a = 50$ and $d = 20$.

Since he has to pay a total of \$$1{,}920.00$, you have to find the number of terms that will sum up to \$$1{,}920.00$.

Given: $a = 50, d = 20, S_n = 1920$:

S_n	$=$	$\frac{1}{2}n[2a + (n-1)d]$	
$1{,}920$	$=$	$\frac{1}{2}(n)[2(50) + (n-1)(20)]$	Substitute n, a, and d
$1{,}920$	$=$	$\frac{n}{2}(100 + 20n - 20)$	Clear inside brackets
192	$=$	$\frac{n}{2}(10 + 2n - 2)$	Divide throughout by 10
192	$=$	$5n + n^2 - n$	
192	$=$	$n^2 + 4n$	
0	$=$	$n^2 + 4n - 192$	Form a quadratic equation
0	$=$	$(n + 16)(n - 12)$	
n	$=$	-16 or 12	Solve the quadratic equation

He will take 12 months to repay the loan, since 12 is the positive answer. We can't have negative number of months.

Practice

1. Sorie starts a job with a monthly salary of Le $500{,}000.00$ with an increase of Le $20{,}000.00$ at the end of each month. What will be his salary at the end of the 10^{th} month?
2. A boy deposits money each day in a cash box for 12 days. He decides to deposit Le $5{,}000.00$ the first day, Le $7{,}000.00$ the second day, Le $9{,}000.00$ the third day and so on.
   a. How much did he deposit on the tenth day?
   b. What will be the total amount he deposits for the 12 days?
3. People are seated in a video centre by rows. The first row has 40 people, the second row 35 people, the third row 30 and so on.
   a. Find the number of people in the sixth row
   b. If there are seven rows in the centre, find the total number of people in the video centre.
4. A child wishes to build up a triangular pile of toy bricks so as to have 1 brick in the top row, 2 in the second row, 4 in the third row and so on. Find:
   a. The number of bricks in the fifth row.
   b. If he built seven rows, how many bricks did he use altogether?
5. A carpenter decides to put sticks in groups. The first group has 1 stick, the second 3, the third 9 and so on. If the last group has 81 sticks, how many groups did he form?

Lesson Title: Ratios	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L005	**Class:** SSS 4

Learning Outcomes

By the end of the lesson, you will be able to:

1. Increase and decrease quantities in a given ratio.
2. Solve real-life problems involving ratio.

Overview

This lesson provides a review of the basic principles of ratio. Ratios are used to compare quantities of the same type, for example: length, weight, people, money and many more.

Ratios can be described in 2 different ways. This is best described using an example. Suppose in an SSS 3 class there are 24 girls and 36 boys. This can be written as:

24 **:** 36	Use a colon to separate the 2 quantities being compared. Read as “24 to 36”.
$\frac{24}{36}$	Write as a fraction

It does not matter which of the 2 ways we write a ratio, we should always simplify it to its lowest terms. This is done by dividing by common factors.

Simplifying the example above gives: 2 : 3 or $\frac{2}{3}$.
This means that for every 2 girls there are 3 boys.

The order in which ratios are written is very important and must be maintained when solving a problem. A ratio written as 2 : 3 means $\frac{2}{3}$, while a ratio written as 3 : 2 means $\frac{3}{2}$.

We can only simplify ratios when the quantities are in the same units. If the quantities are not in the same unit, we must convert one to the other before we simplify

For example, consider a problem: Simplify 5 cm to 5 m.

First, convert 5 metres to centimetres: 5 m = 500 cm.
Rewrite the ratio with both terms in centimetres, and simplify:

5 cm	to	500 cm	
5	**:**	500	Same units, so can be omitted
1	**:**	100	For every 1 cm of one quantity there are 100 cm of another.

We are sometimes required to **increase or decrease** quantities by a given ratio to find the new amounts. The calculation to increase or decrease a quantity Q by a ratio $m : n$ is given by: $\frac{m}{n} \times Q$. An example showing how to increase a quantity in a given ratio is shown in Solved Example 2 below.

We are often asked to **compare 2 or more ratios** to find out which is the biggest or smallest relative to the others. One way we can compare ratios is by writing them as unit ratios.

If we have a ratio in the form $m : n$, we can write it either as $m : 1$ or $1 : n$.

- To write as $m : 1$, we divide both ratios by n.
- To write as $1 : n$, we divide both ratios by m

Once we have converted the given ratios to unit fractions, we can then determine which ratio is greatest or smallest in relation to the others.

A second way to compare ratios is to use LCM to convert each ratio into an equivalent fraction. Both fractions will then have the same denominator. As before, inspect the numerators and determine which ratio is greatest or smallest in relation to the others.

Solved Examples

1. Express $24 : 14$ as a fraction in its lowest terms. If the result is equal to the ratio $x: 35$, find x.

   Solution:

   Step 1. Assess and extract the given information from the problem.

   Given: $24 : 14$; $x : 35$

   Step 2. Change the ratio to its fraction form.

   $\frac{24}{14} = \frac{12}{7}$ Original ratio in lowest terms

   Step 3. Make the given ratios equal.

   $$\frac{12}{7} = \frac{x}{35}$$

   Step 4. Make x the subject of the equality:

   $$\frac{35 \times 12}{7} = x$$

   Step 5. Solve for x. The answer is $x = 60$.

2. Increase Le 60,000.00 in the ratio 8 **:** 5.

   Solution:

   To increase in the ratio 8 **:** 5 means that every Le 5.00 is increased to Le 8.00. We know it is an increase because the first part of the ratio is larger than the second part of the ratio.

New amount $= \frac{8}{5} \times 60{,}000$ Change the ratios to their fraction forms.

$= 96{,}000$ This is a reasonable result as we know the amount increased from before.

The new amount is Le 96,000.00

3. The length and width of a rectangular plot 12 m by 8 m are decreased in the ratio 4:3. Find the ratio by which the area is decreased.
   Solution:
   To find the ratio by which the area is decreased, we will need to find both the original and new area.
   Step 1. Calculate original area: $A = 12 \times 8 = 96 \text{ m}^2$
   Step 2. Calculate the new length and width. Since they are decreased, multiply by the fraction $\frac{3}{4}$, where the numerator is smaller than the denominator:
   New length: $\frac{3}{4} \times 12 \text{ m} = 9 \text{ m}$
   New width: $\frac{3}{4} \times 8 \text{ m} = 6 \text{ m}$
   Step 3. Calculate the new area: $9 \times 6 = 54 \text{ m}^2$
   Step 4. Calculate the ratio by which the area is decreased:
   $96 \text{ m}^2 : 54 \text{ m}^2 = 16 : 9$

4. Express the ratios $3 : 8$ and $4 : 15$ in the form $m : 1$. Which ratio is greater? Use LCM to confirm your result.
   Solution:
   There are 2 methods for solving this problem. Either create unit fractions, or make the denominators equal using the LCM. Both methods are shown.

   Method 1.
   Simplify each ratio independently to a unit fraction

   $\frac{3}{8} = \frac{0.375}{1}$ divide numerator and denominator by 8

   $\frac{4}{15} = \frac{0.267}{1}$ divide numerator and denominator by 15

   Now, compare the 2 ratios.

   $3 : 8 = 0.375 : 1$

   $4 : 15 = 0.267 : 1$

   $3 : 8 > 4 : 15$ since $0.375 > 0.267$

   Method 2.
   Find the LCM of the 2 fractions

   $\frac{3}{8} = \frac{45}{120}$ since the LCM of 8 and 15 is 120

   $\frac{4}{15} = \frac{32}{120}$

   Now, compare the 2 ratios.

   $3 : 8 = 45 : 120$

   $4 : 15 = 32 : 120$

$3 : 8 \quad > \quad 4 : 15$ since $45 > 32$

Answer: $3 : 8$ is the greater ratio.

5. Find which of the pair of ratios is greater:

   a. 2 : 5 or 4 : 7 b. 9 : 13 or 3 : 4

   Solutions:

   Use either method described above. In the examples below, method 2 is used.

   a. Since the LCM of 5 and 7 is 35, rewrite each ratio as a fraction with this as a denominator:

   $2 : 5 = \frac{2}{5} = \frac{14}{35}$

   $4 : 7 = \frac{4}{7} = \frac{20}{35}$

   Compare the fractions. Since $\frac{20}{35} > \frac{14}{35}$, the answer is $4 : 7 > 2 : 5$

   b. Since the LCM of 13 and 4 is 52, rewrite each ratio as a fraction with this as a denominator:

   $9 : 13 = \frac{9}{13} = \frac{36}{52}$

   $3 : 4 = \frac{3}{4} = \frac{39}{52}$

   Compare the fractions. Since $\frac{39}{52} > \frac{36}{52}$, the answer is $3 : 4 > 9 : 13$

Practice

1. A rectangle 20 cm by 10 cm has its length decreased in the ratio 5:4 and its width increased in the ratio 2:3. Find:
   a. the new length and width
   b. the ratio in which the area has changed
2. Increase the following amounts according to the ratios given.
   a. a mass of 60 kg in the ratio 3:7
   b. the price of an article costing Le 2,750 in the ratio 5:8
   c. a length of 2.52 km in the ratio 7:12
   d. a crop of 70 mangoes in the ratio $\frac{1}{2} : 1\frac{1}{2}$
3. Decrease the following amounts according to the ratios given.
   a. 120 m in the ratio 5:2
   b. 1 hour 30 minutes in the ratio 9:4
   c. 6.5 litres in the ratio 13:7
   d. a class of 75 pupils in the ratio $\frac{10}{9} : \frac{2}{3}$
4. Find which of the following pairs of ratios is greater.

   a. 7 : 15 or 8 : 17 b. 11 : 6 or 13 : 7 c. 11 m : 13 m or 7 g : 8 g

   d. Le 170.00 : Le 90.00 or Le 300.00 : Le 60.00

Lesson Title: Rates	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L006	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve problems related to rate, including real-life applications (e.g. rates of pay, travel rates, currency exchange rates).

Overview

Recall that we use ratios to compare two or more "like" quantities. This means that they are of the same kind, e.g. height, temperature, mass, or weight. The quantities must be expressed in the same unit for them to be compared.

We use **rates** when we want to compare quantities of different kinds. For example, how far a motor bike travels in kilometres for a particular length of time in hours, or how much money someone is paid per month at their job.

The quantities in a ratio are measured with one unit. When we write the ratio as a fraction, the units in the ratio cancel each other out because they are the same, e.g. $\frac{\text{area of square}}{\text{area of triangle}} = \frac{\text{cm}^2}{\text{cm}^2}$.

The quantities in the rate are measured with 2 different units. The units in a rate take on the unit from the numerator and the unit from the denominator, e.g. $\frac{\text{distance}}{\text{time}} = \frac{\text{km}}{\text{hr}}$.
Solved Example 1 uses a method called the **unitary method**. The basic procedure is given below.

- We first find the unit rate. This is the amount of the first quantity for every 1 of the second quantity.
- In Example 1, this is 80 kilometres for every 1 hour, i.e. 80 km/hr.
- We then use the unit rate to find all other amounts of one quantity given the other.

A unit rate is a ratio that tells us how many units of one quantity there are for every one unit of the second quantity. Rates use words and symbols such as "per" (/), "each" (ea) and "at" (@).

Solved Examples

1. A car travels a distance of 240 km in 3 hours.
   a. What is the average speed in kilometres per hour (km/hr.)?
   b. How far will it travel in 5 hours?

   Solutions:

   a. Convert to a fraction and simplify to a unit rate.

   240 km **:** 3 hrs Write as a ratio

rate	$=$	$\frac{240 \text{ km}}{3 \text{ hrs}}$	Write as a fraction
	$=$	$\frac{80 \text{ km}}{1 \text{ hr}}$	Write in the form $m : 1$ by dividing numerator and denominator by 3
	$=$	80 km/hr	Write as a rate in km/hr.

The average speed is 80 km/hr.

b. Multiply the unit rate by the number of hours.

speed	$=$	80km/hr	
	$=$	$\frac{80 \text{ km}}{1 \text{ hr}}$	80 km in 1 hr
Distance in 5 hours	$=$	$\frac{80 \text{ km}}{1 \text{ hr}} \times 5$ hrs	Multiply by 5 for distance travelled in 5 hours
	$=$	400 km	The hours cancel each other

The distance travelled in 5 hours is 400 km.

2. Water flows from a pipe into an empty container with a square base with side lengths of 2 cm, at a rate of 64 cm^3 per second. Find the height of the water in the container after 30 seconds.

Solution:

Step 1. Find the total volume that flows into the container in 30 seconds:

In problems involving rate, the relation (formula) is obtained from the units of the rate. In this case the given rate is $64 \frac{\text{cm}^3}{\text{second}} = \frac{64 \text{ cm}^3}{1 \text{ sec}}$. This can be set equal to $\frac{\text{volume}}{\text{time}}$ and used to solve for either volume or time.

Rate	$=$	$\frac{volume}{time}$	From the units of rate in the question
$\frac{64 \; cm^3}{1 \; sec}$	$=$	$\frac{volume}{30 \; sec}$	Substitute the given values
$Volume$	$=$	$64 \times 30 \text{ cm}^3 = 1{,}920 \text{ cm}^3$	Solve for volume

Step 2. Find the height of the water. Use the formula for volume of a rectangular prism.

Volume of container= Area of base × height

$$1920 = 2 \times 2 \times h$$
$$h = 480 \text{ cm}$$

The height of water in the container is 480 cm.

3. Each month, a person works from 8.00 am to 12.30 pm for six days and from 2.00 pm to 5.30 pm for four days. The rate of pay is Le 4,800.00.00 per hour. What is the person‘s total monthly wages?

Solution:

We are given the hourly rate, which is a unitary ratio of $\frac{\text{Le } 4{,}800.00}{1 \text{ hour}}$. Multiply this by the number of hours worked to find the monthly wages.

Step 1. Calculate the hours worked on given days:

$8.00 - 12.30$ = 4 hours 30 min

$2.00 - 5.30$ = 3 hours 30 min

$4 \text{ hours } 30 \text{ min} \times 6$ days = 27 hours

$3 \text{ hours } 30 \text{ min} \times 4$ days = 14 hours

Step 2. Calculate the total hours worked: $27 + 14 = 41$ hours

Step 3. Calculate the total monthly wages: $\text{Le } 4{,}800.00 \times 41 = \text{ Le } 196{,}800.00$

4. Zainab earns $\text{Le } 2{,}000.00$ per hour. What is her overtime pay per hour
   a. At "time and three quarters"
   b. At "double time"

   Solution:

   Overtime is paid at a higher rate per hour. For example, "Time and a half" or "time and a quarter". "Time and three quarters" means that for each hour worked, you are paid for $1\frac{3}{4}$ hours work.

   a.

   $$\begin{aligned} \text{Overtime} &= 1\tfrac{3}{4} \times 2{,}000.00 \qquad \text{Multiply rate by } 1\tfrac{3}{4} \\ &= \tfrac{7}{4} \times 2000 \\ &= \text{Le } 3{,}500.00 \end{aligned}$$

   b.

   $$\begin{aligned} \text{Overtime} &= 2 \times \text{Le } 2000.00 \qquad \text{Double time} \\ &= \text{Le } 4{,}000.00 \end{aligned}$$

5. Three boys can weed a piece of land in 4 hours. How long would it take 18 boys to weed the piece of land, if they weed at the same rate?

   Solution:

   We need to first find the unit rate at which 1 boy weeds land. Then, we apply it to the case where there are 18 boys.

   Step 1. Calculate how long it takes 1 boy to weed land:

   $$\begin{aligned} 3 \text{ boys} &= 4 \text{ hours} \\ 1 \text{ boy} &= 3 \times 4 \text{ hours} \\ &= 12 \text{ hours} \end{aligned}$$

   Step 2. To find how long it will take 18 boys to weed the land, divide 12 hours by 18:

   $$\begin{aligned} 18 \text{ boys} &= \tfrac{12}{18} \text{ hours} \\ &= \tfrac{2}{3} \text{ hours} \\ &= \tfrac{2}{3} \times 60 \text{ min} \\ &= 40 \text{ min} \end{aligned}$$

Practice

1. A vehicle consumes 180 litres of fuel for a distance of 650 km. How many litres of fuel will be consumed for a distance of 1,625 km?
2. A piece of work takes 20 labourers to complete in 10 days. How many labourers will complete the same job in 25 days if they work at the same rate?
3. A group of 15 boys can load a truck in 4 hours. How long will it take the following groups to load the same truck working under the same conditions?
    a. 5 boys
    b. 30 boys
4. A car uses petrol at the rate of 1 litre for every 11 km. If the price of petrol is Le 6,000.00 per litre, find the cost of the petrol for a journey of 891 km.
5. In the year 2000, a factory produced 9,324 bicycles. Allowing 2 weeks for holidays and a further 100 days for weekends, find the rate of production in bicycles per day.
6. Amadu earns $\text{Le } 6{,}000.00$ per hour. What is his overtime pay per hour:
    a. At "time and two thirds"
    b. At "triple time"
7. A workman is paid Le 152,000.00 for a 40-hour week. Calculate his hourly rate of pay.
8. A car travels 153 km in $2\frac{1}{4}$ hours. Calculate its average speed in km/h.
9. Water flows from a pipe into an empty cylindrical container at the rate of 36π cm^3 per second. If the radius of the container is 3 cm, find the height of the water in the container after 8 seconds.
10. The population density of a town is 400 people/km^2.What is the population of the town if the area is 2.5 km^2?

Lesson Title: Proportional division	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L007	**Class:** SSS 4

Learning Outcome

By the end of the lesson, you will be able to divide quantities into given proportions, and solve real-life applications.

Overview

This lesson is a review of proportional division. We are familiar with doing calculations where we share quantities, for example money, equally between several people. Suppose we are asked instead to share the same amount of money according to a given ratio. We perform a **proportional division** according to the given ratio.

A quantity shared equally will result in the same amount per share. A quantity shared in different proportions will result in different amounts per share according to the given ratio.

To share something in a given ratio, find the total number of parts in that ratio. This is done by adding all of the numbers in the ratio. Then, find one part of the whole by multiplying by a fraction with the total number of parts in the denominator.

Solved Examples

1. Share Le 750,000.00 between 2 children at the ratio 8 : 7. How much will each child receive?
   Solution:
   Step 1. Find the total number of parts to the ratio: $8+7=15$
   Note that the ratio 8 : 7 means that for every Le 15.00 of the amount to be shared, Le 8.00 will go to Child 1 and Le7.00 will go to Child 2
   Step 2. Find what proportion (fraction) of the total is given to each part.
   Child 1 receives: $\frac{8}{15}\times 750{,}000=400{,}000$
   Child 2 receives: $\frac{7}{15}\times 750{,}000=350{,}000$

   Answer: Child 1 receives Le 400,000.00. Child 2 receives Le 350,000.00.

2. The weights of two pets are in the ratio 3: 5. The heavier pet is 10 kg. What is the weight of the lighter pet?
   Solution:
   Step 1. Find the total number of parts to the ratio: $3+5=8$
   Step 2. Let k be the total weight of the 2 pets. Then the heavier pet weighs $\frac{5}{8}\times k$ and the lighter pet weighs $\frac{3}{8}\times k$. Set up an equation with the heavier pet's weight and solve for k:

$$\frac{5}{8} \times k = 10 \text{ kg}$$
$$5k = 8 \times 10 \text{ kg}$$
$$\frac{5k}{5} = \frac{80}{5}$$
$$k = 16 \text{ kg} \quad \text{Total weight of the 2 pets}$$

Step 3. Find the weight of the lighter pet:

$$\text{weight of lighter pet} = \frac{3}{8} \times 16 \text{ kg}$$
$$= 6 \text{ kg}$$

3. If 3 boys shared x oranges among themselves in the ratio 3 : 5 : 8 and the smallest share was 45 oranges, find the value of x.

   Solution:

   Step 1. Find the total number of parts to the ratio: $3 + 5 + 8 = 16$

   Step 2. Set up an equation and solve for x:

$$\frac{3}{16} \times x = 45 \text{ oranges} \quad \text{smallest share}$$
$$3x = 16 \times 45$$
$$3x = 720$$
$$\frac{3x}{3} = \frac{720}{3}$$
$$x = 240$$

4. The expenditure on education, health and water supply are in the ratio 7:11:5. If the expenditure on health is Le 22,000,000.00, find:
   a. The total expenditure for education, health, and water.
   b. The expenditure for education.
   c. The expenditure for water supply.

   Solution:

   a. Let E be the total expenditure.

   Step 1. Find the total number of parts to the ratio: $7 + 11 + 5 = 23$

   Step 2. Set up an equation and solve for E:

$$\frac{11}{23} \times E = \text{Le } 22{,}000{,}000.00$$
$$11E = 23 \times 22{,}000{,}000$$
$$11E = \text{Le } 506{,}000{,}000.00$$
$$\frac{11E}{11} = \frac{Le\ 506{,}000{,}000.00}{11}$$
$$E = \text{Le } 46{,}000{,}000.00$$

   The total expenditure is Le 46,000,000.00

   b. Let e be the expenditure for education.

$e \quad = \frac{7}{23} \times \ 46{,}000{,}000$
$\quad\;\; =$ Le $14{,}000{,}000.00$

c. Let w be the expenditure for water supply.

$w \quad = \frac{5}{23} \times \ 46{,}000{,}000$
$\quad\;\; =$ Le $10{,}000{,}000.00$

Practice

1. In a bag of oranges, the ratio of the good ones to the bad ones is 5 : 4. If the number of bad oranges in the bag is 36, how many oranges are there altogether?
2. The ratio of men to women in a church is 5 : 7. If there are 1,200 people in the church, how many men are there?
3. If 3 boys shared x oranges among themselves in the ratio 3 : 5 : 8 and the smallest share was 45 oranges, find the value of x.
4. The ratio of pocket money received by three friends, Momodu, Musa and Amadu, is 5 : 2 : 7. If Amadu gets Le 1,400.00 pocket money, find how much does Momodu and Musa receive?
5. The cost Le 232,000.00 of producing a machine arises from the cost of materials, labour and overhead in the ratio 7 : 9 : 2. Calculate the cost of labour for producing 32 such machines.
6. Fatu and Adama invested Le 1,200,000.00 and Le 1,800,000.00 respectively in a business. If they agree to share any profit in proportion to their investments, what is the share of each for a profit of Le 300,000.00?
7. Abdul, Karim and Unisa are 7 years, 8 years and 10 years old respectively. If Le 2,000,000.00 is shared amongst them in the ratio of their ages, how much does each receive?

Lesson Title: Speed	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L008	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve problems involving speed, time, and distance.

Overview

The formula connecting speed, distance and time is given by $d = st$, where d Is the distance travelled, s is the speed and t is the time taken to cover the distance.

The other 2 formulas are easily derived from the above: $s = \frac{d}{t}$ and $t = \frac{d}{s}$.
Use these formulas whenever a problem asks "how fast", "how far", or "how long".

The speed s can be defined either as a constant speed over a particular distance or the average speed for a journey. If it is average speed it is given by:
average speed $= \frac{\text{total distance travelled}}{\text{total time taken}}$

There are times when the connection between distance, speed and time leads to more complex equations. They may lead to simultaneous linear equations, or a quadratic equation that needs to be solved.

Solved Examples

1. Mr. Kotey left his office at Kaneshie in his car at 5:30 pm to his house at Abeka at a steady speed of 72 km/h. If he arrived at his house at 5:35 pm, find the distance between his office and the house.
   Solution:
   Step 1. Calculate time.
   Time in minutes: $5.35 \text{ pm} - 5.30 \text{ pm} = 5$ minutes;
   Convert minutes to hours: $5 \times \frac{1 \text{ hour}}{60 \text{ minutes}} = \frac{1}{12}$ hours
   Step 2. Apply the formula for distance
   $$\begin{aligned} d &= s \times t \\ &= 72 \times \frac{1}{12} \\ &= 6 \text{ km} \end{aligned}$$
   Answer: The distance between his office and the house is 6 km.

2. An Okada driver covered half the distance between two towns in 2 hr 30 min. After that he increased his speed by 2 km/hr. He covered the second half of the distance in 2 hr 20 min. Find:
   a. The initial speed of the driver.
   b. The distance between the two towns.

Solutions:

a. **Step 1.** Assess and extract the given information from the problem

Given: 2 part journey: 1st part: 2 hr 30 min at speed x km/hr

2nd part: 2 hr 20 min at speed $x+2$ km/hr

total distance travelled = $2y$ km

where x is the initial speed and y is half the distance between the towns.

Step 2. Substitute into the appropriate formula.

$$\text{distance} = \text{speed} \times \text{time}$$

1st part: $y = x \times 2\frac{1}{2}$ since 2 hr 30 min $= 2\frac{1}{2}$ hr

$= \frac{5}{2}x$

2nd part: $y = (x+2) \times 2\frac{1}{3}$ since 2 hr 20 min $= 2\frac{1}{3}$ hr

$= \frac{7}{3}(x+2)$

since the distances are equal:

$$\frac{5}{2}x = \frac{7}{3}(x+2)$$
$$= \frac{7}{3}x + \frac{14}{3}$$
$$\left(\frac{5}{2} - \frac{7}{3}\right)x = \frac{14}{3}$$
$$\frac{1}{6}x = \frac{14}{3}$$
$$x = \frac{14 \times 6}{3} = 28 \text{ km/hr}$$

The initial speed of the driver = 28 km/hr.

b. distance $y = \frac{5}{2}x$

$= \frac{5}{2} \times 28$

$=$ 70 km (70 km is halfway, 70 x 2 = 140)

The distance between the 2 towns is 140 km.

3. A motorist drove from town P to town Q, a distance of 80 km, in 30 minutes.
   a. What is his average speed in km/h?
   b. What is his average speed in m/s?

Solutions:

a. Convert minutes to hours, then apply the formula for speed.

Step 1. Convert time: $t = 30\text{ min} = \frac{1}{2}\text{ hour}$

Step 2. Apply the formula:

$$s = \frac{d}{t}$$
$$= \frac{80}{\frac{1}{2}}$$
$$= 80 \times \frac{2}{1}$$
$$= 160\text{ km/h}$$

b. Convert kilometres to metres and time to seconds, then apply the formula for speed.

Step 1. Convert distance: $d = 80\ km = 80\ \times 1{,}000 = 80{,}000\ \mathrm{m}$

Step 2. Convert time: $t = 30\ \mathrm{min} = 30 \times 60 = 1{,}800\ sec.$

Step 3. Apply the formula:

$$\begin{aligned} s &= \frac{d}{t} \\ &= \frac{80{,}000}{1{,}800} \\ &= 44.4\ \mathrm{m/s} \end{aligned}$$

4. A bus and a poda-poda both left the bus terminal at the same time heading in the same direction. The average speed of the poda-poda is 30 km/hr slower than twice the speed of the bus. In two hours, the poda-poda is 20 miles ahead of the bus. Find: a. The speed of the bus. b. The speed of the poda-poda.

Solutions:

a. **Step 1.** Assess and extract the given information from the problem.

Given: Bus and poda-poda leave bus terminal together in the same direction. Poda-poda driving 30 km/hr slower than twice the speed of the bus, and is 20 miles ahead of the bus after 2 hours.

Step 2. Assign variables to the unknown quantities

distance of bus	=	d	distance of poda-poda	=	$d + 20$
speed of bus	=	s	speed of poda-poda	=	$2s - 30$

Step 3. Set up the equations.

distance	=	speed × time		
d	=	$2s$	(1)	$t = 2$ hours
$d + 20$	=	$2(2s - 30)$	(2)	same time

We now have 2 linear equations in d and s

Solve simultaneously by substitution

Step 4. Substitute equation (1) into equation (2) and simplify:

$$\begin{aligned} 2s + 20 &= 2(2s - 30) \\ &= 4s - 60 \\ 80 &= 2s \\ s &= 40 \text{ km/hr} \end{aligned}$$

Step 5. Write the speed of the bus.

The speed of the bus is 40 km/hr.

b. **Step 6.** Find the speed of the poda-poda.

$$\begin{aligned} \text{speed of poda-poda} &= 2s - 30 \\ &= (2 \times 40) - 30 \\ &= 80 - 30 = 50 \text{ km/hr} \end{aligned}$$

The speed of the poda-poda is 50 km/hr.

Practice

1. Find the distance in km travelled by an airplane moving at 100 m/s for 15 minutes.
2. A motorist covers a distance of 600 m in 10 minutes; find his average speed in kilometres per hour.
3. A man covered a distance of 10 km in 45 minutes on his bicycle. Find his speed in kilometres per hour. Give your answer to 3 significant figures.
4. A boy cycles 15 kilometres to school in 55 minutes. Find his average speed in meters per second. Give your answer to 1 decimal place
5. An athlete ran 1.5km in 4 minutes 10 seconds. What was his average speed in metres per second? Give your answer to 1 decimal place.
6. A car is travelling at an average speed of 80km/h. What is its speed in metres per second (m/s) Give your answer to 2 decimal places.
7. A car moving at an average speed of 60km/h takes 3 hours to cover a certain distance. How far is the journey of the car?
8. A bus travelled 40km at a speed of 10km/h. If the bus made the return journey at an average speed of 40km/h, find the average speed of the whole journey.

Lesson Title: Applications of percentages – Part 1	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L009	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve problems involving commission, income taxes, simple interest and compound interest.

Overview

Commission

Some employees, particularly sales people, are given commission on top of (or instead of) their wages or salaries. The value of the commission is usually worked out as a percentage of the amount they sold during the month or year.

The value of the amount sold is taken as 100%.
To calculate x commission on a particular sales amount, use the formula:

$$\text{commission} = \frac{x}{100} \times \text{sales amount}$$

Income Taxes

Income tax is deducted every month by the government from the money people earn. It is used to provide services to the country such as education, health, police, military and social welfare.

Employee taxes are deducted from their salaries by their employers using a method called PAYE. PAYE stands for Pay As You Earn and the 2017 rates are shown on the table on the board.

Sierra Leone PAYE Tax Rate	
Not over Le 500,000 per month	Nil
Next Le 500,000 per month	15%
Next Le 500,000 per month	20%
Next Le 500,000 per month	30%
Above Le 2 million per month:	35%

Every employee has a tax-free income. This is the income below which you do not have to pay any income tax. The current tax-free income is Le 500,000.00.
The net income an employee earns is the income after tax has been deducted.

Interest

When someone deposits money in a bank, the bank pays them interest on the money deposited. When a bank lends money to its customers, it charges them interest on the money borrowed.
There are two types of interest earned or charged on money – **simple interest** and **compound interest.**
The **simple interest,** I**,** is the amount earned or charged on the initial amount or principal, P, at a given rate, R, and for a given period of time, T (in years).

$$I = \frac{PRT}{100}$$

It is in effect the percentage of the principal that is earned or charged for the use of the money.
The amount, *A*, at the end of the period is given by Principal + Interest. That is:

$$A \quad = \quad P + I$$

Compound interest is the interest calculated at given intervals over the loan period and added to the principal. This new amount becomes the principal and changes every time the interest is calculated. Each time we do the calculation, we compound the principal by adding the interest calculated for a given period to the previous principal.

We are in effect earning or paying interest on the interest.

Each period is called a **compounding period** and can be at intervals of 1 year, 6 months ($\frac{1}{2}$ year), 3 months ($\frac{1}{4}$ year) or any other agreed time period.

The compound interest, CI, is given by:

$CI \quad = \quad A - P$ **where** $A =$ Amount at the end of the period
$P =$ Principal

We will now do an example to show 2 different methods of calculating compound interest. We will concentrate on calculating the compound interest annually.

Solved Examples

1. A newspaper vendor makes a commission of 12% on his sales. Calculate his commission on the following sales: a. Le $6{,}000.00$ b. Le $340{,}000.00$

   Solution:

   Step 1. Assess and extract the given information from the problem.
   Given: commission received by sales vendor = 12% of sales

   Step 2. Calculate commission for each sales amount.

   commission $= \frac{12}{100} \times$ sales amount $= 0.12 \times$ sales amount

   a. Le 6,000.00 sales: commission $= 0.12 \times 6{,}000 =$ Le 720.00

   b. Le 340,000.00 sales: commission $= 0.12 \times 340{,}000 =$ Le $40{,}800.00$

2. Jenneh gets a commission of 10% on bread sold. In one week, Jenneh's commission was Le $45{,}000.00$. How much bread did she sell during that week?

   Solution:

   Given: 10% commission on bread sold; Jenneh's commission was Le $45{,}000.00$

   Let amount of sales $= x$

   commission $= \frac{10}{100} \times x = 0.1x$

   $45{,}000 = 0.1x$

   $x = \frac{45{,}000}{0.1}$

   $x =$ Le $450{,}000.00$

   Jenneh sold Le$450{,}000.00$ worth of bread.

3. Use the table of income tax rates to calculate how much tax is paid on a salary of Le 850,000.00 each month.

   Solution:

   Step 1. Assess and extract the given information from the problem.
   Given: Income tax for salary of Le 850,000.00
   Le 500,000.00 is tax-free (from the table)

Step 2. Calculate the income tax paid per month on Le 850,000.00 salary

taxable income $= 850{,}000 - 500{,}000 =$ Le $350{,}000.00$

income tax $= \frac{15}{100} \times 350{,}000 =$ Le $52{,}500.00$

Step 3. Write the answer.

The income tax is Le $52{,}500.00$ per month.

4. Sama pays income tax of Le 187,000.00 each month. How much does he earn per month?

Solution:

Given: income tax of Le 187,000.00 each month

We do a reverse calculation to the solved example above.

Remaining tax (Le)	Amount to be taxed (Le)	Rate of tax (%)	Income Tax
$187{,}000$	$500{,}000$	15	$\frac{15}{100} \times 500{,}000 = 75{,}000$
$187{,}000 - 75{,}000 = 112{,}000$	$500{,}000$	20	$\frac{20}{100} \times 500{,}000 = 100{,}000$
$112{,}000 - 100{,}000 = 12{,}000$	x	30	$\frac{30}{100} \times x = 0.3x$

From the amount of tax remaining in the last line, we know that the amount to be paid is less than Le 500,000.00. Let the amount $= x$, such that:

$0.3x = 12{,}000$ income tax = remaining tax

$x = \frac{12{,}000}{0.3} =$ Le $40{,}000.00$

taxable income $= 500{,}000 + 500{,}000 + x$ from the table

$= 500{,}000 + 500{,}000 + 40{,}000 = 1{,}040{,}000$

income = tax-free income + taxable income

$= 500{,}000 + 1{,}040{,}000 = 1{,}540{,}000$

Sama earns Le $1{,}540{,}000.00$.

5. Alusine deposits Le 500,000.00 in the bank at the rate of 4% per annum for 2 years. How much interest does he receive?

Solution:

Step 1. Assess and extract the given information from the problem.

Given: Le 500,000.00 deposited by Alusine 4% interest rate per annum for 2 years

Step 2. Calculate the interest.

$$I = \frac{PRT}{100}$$

$$I = \frac{500{,}000 \times 4 \times 2}{100} = \text{Le } 40{,}000.00$$

The interest received by Alusine is Le $40{,}000.00$.

6. Find the time period in which the interest on Le 300,000.00 at 3% interest rate is Le 45,000.00.

Solution:

Given: interest on Le 300,000.00 at 3% interest rate = Le 45,000.00.

$T \quad = \quad \frac{I \times 100}{PR}$ make T the subject of the formula

$T \quad = \quad \frac{45{,}000 \times 100}{300{,}000 \times 3} \quad = \quad$ 5 years

The time period $= 5$ years.

7. A business woman deposited Le 3,000,000.00 in her bank account at 7% compound interest rate per annum for 5 years. At the end of the third year, she withdrew Le 1,000,000.00. Calculate the amount she has in her account after 5 years. Give your answer to the nearest cent.

Solution:

Given: Businesswoman deposits Le 3,000,000.00 at 7% rate per annum for 5 years.

Multiplier $= 1 + \frac{7}{100} = 1.07$

Year	Principal at start of year (Le)	Amount at end of year (Le)	
1	$3{,}000{,}000$	$3{,}000{,}000 \times 1.07$	$= 3{,}210{,}000$
2	$3{,}210{,}000$	$3{,}210{,}000 \times 1.07$	$= 3{,}434{,}700$
3	$3{,}434{,}700$	$3{,}434{,}700 \times 1.07$	$= 3{,}675{,}129$
	withdrawal of Le 1,000,000: new principal		$= 3{,}675{,}129 - 1{,}000{,}000$
			$=$ Le $2{,}675{,}129.00$
4	$2{,}675{,}129$	$2{,}675{,}129 \times 1.07$	$= 2{,}862{,}388.03$
5	$2{,}862{,}388.03$	$2{,}862{,}388.03 \times 1.07$	$= 3{,}062{,}755.19$

After 5 years, the businesswoman has Le $3{,}062{,}755.19$ to the nearest cent in her account.

Practice

1. A newspaper vendor makes a commission of 12% on his sales. Calculate his commission on the following sales: a. Le 6,000.00 b. Le 340,000.00
2. Every month, a sales agent selling electrical goods makes commission of 3% on the first Le 2 million of sales, 4% on the next Le 3 million of sales and 5% on any sales over Le 5 million. How much commission does he make on sales of Le 16 million in December?
3. Use the income tax table to calculate how much tax is paid on a salary of Le 1,700,000.00 each month.
4. Adama earns Le 1,200,000.00 per month. In addition to her tax-free income, she can claim Le 50,000.00 for every dependent child. She has 3 children. Calculate:
   a. Her total tax-free income b. Her taxable income
   c. Her total tax per month d. Her net income per month
5. The simple interest on Le 725,000.00 for 4 years is Le 87,000.00. How much per annum is the interest rate?
6. How much money should be invested if interest of Le 90,000.00 is to be paid after 3 years at 5% per annum? What is the amount after 3 years?
7. Find the simple interest on a loan of Le 500,000 for 4 years at 5% per annum.
8. Find the interest on a loan of Le 500,000 for 4 years at a compound interest rate of 5% per annum. What additional interest is earned using compound as compared to simple interest rate?

Lesson Title: Applications of percentages – Part 2	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L010	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve problems involving profit, loss, hire purchase, and discount.

Overview

Profit and Loss

An item is sold at a profit when the selling price is greater than the cost price of the item. If, however, the cost price of the item is greater than the selling price, the item is sold at a loss. The profit or loss is calculated by taking the difference between the cost price (*CP)* and selling price (*SP).*
Note that the difference is always positive,

$$\text{profit} = SP - CP$$
$$\text{loss} = CP - SP$$

Percentage profit or loss based on the cost price is given by:

$$\text{Percentage profit } = \frac{SP-CP}{CP}\times 100$$

Percentage loss based on the cost price is given by:

$$\text{Percentage loss } = \frac{CP-SP}{CP}\times 100$$

Solved Examples 1-3 are on profit and loss.

Hire Purchase

There are instances when an item is bought and the full amount is paid in regular instalments over several months or years. Paying an item over time usually costs more than the cash price when bought outright. This is because interest is usually added to the price of the item being sold.

A deposit may be paid for the item so that the buyer can use it right away. However, the item does not belong to the buyer until it has been completely paid for. The interest charged can be calculated using the simple interest rate based on the length of the loan.

Solved Examples 4 and 5 are on hire purchase.

Discount

A discount is given in shops when customers buy in bulk or when there is a special offer. The discount is usually given as a percentage of the original price. The original price is 100% or 1 $\left(\frac{100}{100}\right)$. We use a multiplier which is given by $1 - \frac{R}{100}$ where R is the percentage discount.

Solved Examples 6 and 7 are on discount.

Solved Examples

1. John buys a set of bicycle pumps for Le 40,000.00 and sells them for Le 50,000.00. Find his percentage profit.
   Solution:

   $$\textbf{\% profit} = \frac{SP-CP}{CP}\times 100$$
   $$= \frac{50{,}000-40{,}000}{40{,}000}\times 100$$
   $$= \frac{10{,}000}{40{,}000}\times 100$$
   $$= 25\%$$

   John made 25% profit.

2. A man bought a car for Le 15,000,000.00. He later sold it for Le 12,000,000.00. What was his percentage loss on the sale of the car?
   Solution:

   $$\text{percentage loss} = \frac{SP-CP}{CP}\times 100$$
   $$\text{percentage loss} = \frac{15{,}000{,}000-12{,}000{,}000}{15{,}000{,}000}\times 100$$
   $$= \frac{3{,}000{,}000}{15{,}000{,}000}\times 100$$
   $$= 20\%$$

   The percentage loss was 20%.

3. Akin bought a television set at the second-hand shop. He sold it for Le 2,500,000.00. If he made a profit of 25%, how much did he buy the television?
   Solution:

   Method 1. Use the formula for percentage profit.

   $$\text{percentage profit} = \frac{SP-CP}{CP}\times 100$$
   $$25 = \frac{2{,}500{,}000-CP}{CP}\times 100$$
   $$25 = \frac{(2{,}500{,}000-CP)\times 100}{CP}$$

   Multiply throughout by the cost price, CP

   $$25CP = (2{,}500{,}000 - CP)\times 100$$
   $$\frac{25}{100}CP = 2{,}500{,}000 - CP$$
   $$0.25CP + CP = 2{,}500{,}000$$
   $$1.25CP = 2{,}500.000$$
   $$CP = \frac{2{,}500{,}000}{1.25}$$
   $$= \text{Le } 2{,}000{,}000.00$$

   Method 2. Use a multiplier

   $$SP = CP + \frac{25}{100}CP$$

   since Akin made a profit, we add the percentage profit to 100% of the cost price

   $$= CP\left(1+\frac{25}{100}\right)$$
   $$\text{multiplier} = 1 + \frac{25}{100} = 1.25$$
   $$SP = 1.25\times CP$$
   $$2{,}500{,}000 = 1.25CP$$
   $$CP = \frac{2{,}500{,}000}{1.25}$$
   $$= \text{Le } 2{,}000{,}000.00$$

Akin bought the television set for Le $2{,}000{,}000.00$.

4. Mr. Kargbo wants to buy a car on sale at Le 25,000,000.00 cash. He paid Le 5,000,000.00 deposit and 15% simple interest charged on the remainder for 2 years. How much interest did he pay?

Solution:

First, find the remainder that Mr. Kargbo needs to pay:

remainder to be paid $= 25{,}000{,}000 - 5{,}000{,}000 = 20{,}000{,}000$

Calculate the interest on this amount. For T, use the length of the loan (2 years):

$$I = \frac{PRT}{100}$$

$$I = \frac{20{,}000{,}000 \times 15 \times 2}{100}$$

$$= \text{Le } 6{,}000{,}000.00$$

The interest paid by Mr. Kargbo is Le $6{,}000{,}000.00$.

5. A retailer offers the following hire purchase terms on generators: Deposit 30% of the cash price, then 4 monthly instalments charged at a simple interest rate of 20% on the remainder. If the cash price is Le 2,250,000.00, find:
   a. The remainder on which interest is charged,
   b. The monthly instalments.

Solutions:

a. remainder $= \left(1 - \frac{30}{100}\right) \times 2{,}250{,}000 = \text{Le } 1{,}575{,}000.00$

b. average time, $T = \frac{1+4}{2}$ average time is used as explained before

$= \frac{5}{2} =$ months $= \frac{2.5}{12}$ years

$I = \frac{PRT}{100} = \frac{1{,}575{,}000 \times 20 \times 2.5}{100 \times 12}$

$= \text{Le } 65{,}625.00$

total cost over 4 months $= 1{,}575{,}000 + 65{,}625$

monthly instalment $= \frac{1{,}640{,}625}{4}$

$= \text{Le } 410{,}156.25$

Interest is charged on Le $1{,}575{,}000.00$ with 4 monthly instalments of Le $410{,}156.25$.

6. A gas cooker costs Le 1,250,000.00. If the shop offers a customer a 20% discount, how much will the customer pay for the cooker?

Solution:

Calculate how much the customer pays.

multiplier $= 1 - \frac{20}{100}$

$= 1 - 0.2$

$= 0.8$

amount paid $= 0.8 \times 1{,}250{,}000 = \text{Le } 1{,}000{,}000.00$

The customer pays Le $1{,}000{,}000.00$.

7. What percentage discount was given on an item reduced from Le 250,000.00 to Le 212,500.00?

Solution:

$$\text{discount} = 250{,}000 - 212{,}500 = \text{Le } 37{,}500.00$$

$$\text{percentage discount} = \frac{37{,}500}{250{,}000} \times 100$$

$$= 15\%$$

The percentage discount is 15%.

Practice

1. A trader bought 10 boxes of fruit at Le 20,000.00 each. She sold 4 boxes for Le 25,000.00 each, 3 boxes for Le 30,000.00 and the remainder for Le18,000.00 each.
    a. How much profit or loss did the trader make on the boxes of fruit?
    b. What was the average selling price per box?
2. Mrs. Mansaray bought an oven on hire purchase for Le 1,687,500.00. She paid 12.5% more than if she had paid cash for the oven. If she made an initial deposit of 20% of the cash price and then paid the rest in 6 monthly instalments, find:
    a. The initial deposit
    b. The amount of each instalment
    c. The approximate rate of interest to 1 decimal place
3. Amadu bought a motor bike on sale at Le 4,500,000.00 cash. He paid Le 1,500,000.00 deposit and 12% simple interest charged on the remainder for 3 years. How much interest did he pay?
4. A motor bike costs Le 4,500,000.00. The shop gives a discount of $33\frac{1}{3}$% for cash.
    a. How much will a buyer save by paying cash for the motor bike?
    b. How much will the buyer pay for the motor bike?
5. A school buys exercise books from a supplier. He gives the school a 15% discount if they buy more than 500 and a 20% discount for buying over 1,000 books. If each book costs Le 1,500, how much will they save if they buy:
    a. 750 books b. 1,250 books
6. A retailer discounted her prices by 15% for a month. She then gave a further 10% off the discounted price.
    a. How much will an item originally costing Le 48,000.00 now cost?
    b. How much percentage profit will she lose by selling at this price?

Lesson Title: Applications of percentages – Part 3	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L011	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve problems involving depreciation, financial partnerships, and foreign exchange.

Overview

Depreciation

Many goods lose their value over time as they get older. Examples are cars, computers and mobile phones. This decrease in value is called **depreciation**.

In a previous lesson, we looked at calculating compound interest using the formula $A = P\left(1 + \frac{R}{100}\right)^n$. We can use a similar formula to calculate depreciation. With compound interest the value appreciates or increases over time, but with depreciation, it depreciates or decreases over time. For depreciation the percentage rate will be subtracted.
The value at the end of a particular time period is given by:

$$V = P\left(1 - \frac{R}{100}\right)^n$$

where $V =$ Value at the end of the period
$R =$ Rate of depreciation
$P =$ Original price
$n =$ Period

The rate of depreciation can be found using the formula $R = \frac{P-V}{P} \times 100$.

Solved Examples 1 and 2 are on depreciation.

Financial Partnerships

When 2 or more people come together and invest money for the purpose of providing goods or services at a profit, it is called a financial or business partnership. Partnerships are usually formed by professionals such as lawyers, doctors, architects and engineers who wish to pool their resources together. In many instances, the partners pay out profit in proportion to the money or capital invested.

Solved Example 3 is on financial partnerships.

Foreign exchange

Every country has its own currency which it uses for its money. The table to the right gives some countries and currencies. The exchange rate is the rate at which one unit of a particular currency is converted to another currency. There are usually two rates – the **buying** and the **selling** rate.

Country	Currency	Symbol
Ghana	Cedi	GH¢
Gambia	Dalasi	D
Germany	Euro	€
Great Britain	Pounds	£
Nigeria	Naira	₦
Sierra Leone	Leones	Le
United Sates	Dollars	$

The table shows the buying and selling rates in a bank for various currencies on a particular day. Use these rates for problems in this lesson. The bank buys from customers at the buying rate and sells at the selling rate. The selling rate is higher than the buying rate. This allows the bank to make a profit in trading in the currency.

Currency	Buying	Selling
€ 1.00	Le 8,600	Le 8,900
GH¢ 1.00	Le 1,500	Le 1,560
GMD 1.00	Le 150	Le 156
₦ 1.00	Le 20	Le20.80
£ 1.00	Le 9,500	Le 9,800
$ 1.00	Le 7,600	Le 7,900

Solved Examples 4 and 5 are on foreign exchange.

Solved Examples

1. A car costs Le 25,000,000.00 and depreciates at 20% per annum. Find its value after: a. 1 year b. 3 years

   Solutions:

   Apply the formula $V = P\left(1 - \frac{R}{100}\right)^n$.

   a. Its value after 1 year is:

   $$V = 25{,}000{,}000\left(1 - \frac{20}{100}\right)^1$$
   $$= \text{Le } 20{,}000{,}000.00$$

   b. Its value after 3 years is:

   $$V = 25{,}000{,}000\left(1 - \frac{20}{100}\right)^3$$
   $$= \text{Le } 12{,}800{,}000.00$$

2. A motor bike costs Le 4,500,000.00. Its value depreciates by 18% the first year and 15% the second and subsequent years.
   a. What is its value at the end of 5 years? Give your answer to 2 decimal places.
   b. What was the average rate of depreciation over the 5 years?

   Solutions:

   a. Apply the formula $V = P\left(1 - \frac{R}{100}\right)^n$. Note that it will need to be applied twice: once for the first year, and again for the subsequent 4 years.

   After 1 year, $V = 4{,}500{,}000\left(1 - \frac{18}{100}\right)^1 = \text{Le } 3{,}690{,}000.00$

   After 4 more years, $V = 3{,}690{,}000\left(1 - \frac{15}{100}\right)^4 = \text{Le } 1{,}926{,}203.06$

   b. Calculate the average rate of depreciation over 5 years:

   $$\text{Average rate} = \frac{18+15+15+15+15}{5} = \frac{78}{5} = 15.6\%$$

3. Two sisters Kemi and Yemi entered into a business partnerships. Kemi contributed Le 5,600,000.00 and Yemi contributed Le 2,400,000.00. At the end of the year, they made a profit of 70% of their total contribution. A total of 20% of the

profit was reserved for re-investment and 2.5% of the remaining profit was paid into a trust fund for their children. If they shared the remaining profit in the ratio of their contributions, find:

a. The amount reserved for re-investment
b. The amount paid into the trust fund
c. The amount received by each partner as her share of the profit
d. Each sister's share as a percentage of her contribution

Solutions:

a. Calculate the amount reserved for re-investment:

total contribution	=	$5{,}600{,}000 + 2{,}400{,}000$	=	Le $8{,}000{,}000.00$
profit	=	$\frac{70}{100} \times 8{,}000{,}000$	=	Le $5{,}600{,}000.00$
amount reserved for re-investment	=	$\frac{20}{100} \times 5{,}600{,}000$	=	Le $1{,}120{,}000.00$

b. Calculate the amount paid into a trust fund:

remaining profit	=	$5{,}600{,}000 - 1{,}120{,}000$	=	Le4 ,480,000.00
amount paid into a trust fund	=	$\frac{2.5}{100} \times 4{,}480{,}000$	=	Le $112{,}000.00.00$

c. Calculate each partner's share:

remaining profit	=	$4{,}480{,}000 - 112{,}000$	=	Le $4{,}368{,}000.00$
ratio of contribution	=	$\frac{5{,}600{,}000}{2{,}400{,}000}$	=	$\frac{7}{3}$
total number of parts	=	$7 + 3$	=	10
Kemi's share	=	$\frac{7}{10} \times 4{,}368{,}000$	=	Le $3{,}057{,}600.00$
Yemi's share	=	$\frac{3}{10} \times 4{,}368{,}000$	=	Le $1{,}310{,}400.00$

d. Calculate each partner's percentage share:

% share for Kemi	=	$\frac{3{,}057{,}600}{5{,}600{,}000} \times 100$	=	54.6%
% share for Femi	=	$\frac{1{,}310{,}400}{2{,}400{,}000} \times 100$	=	54.6%

4. How much will Le 5,000,000.00 give you in the following currencies? Use the selling rate.

a. US $ b. GB £ c. Gambia D

Solutions:

Use the unitary method and conversion rate to calculate the amount in the required currency.

a.

Le $7{,}900.00$	=	$\$1.00$
Le 1.00	=	$\$\frac{1.00}{7{,}900}$
Le $5{,}000{,}000.00$	=	$\$\frac{1.00}{7{,}900} \times 5{,}000{,}000$
	=	$\$632.91$

b.

$$\text{Le } 9{,}800.00 = £1.00$$
$$\text{Le } 5{,}000{,}000.00 = £\frac{1.00}{9800} \times 5{,}000{,}000$$
$$= £510.20$$

c.

$$\text{Le } 156.00 = \text{Gambian D}1.00$$
$$\text{Le } 5{,}000{,}000.00 = \frac{1.00}{156} \times 5{,}000{,}000$$
$$= \text{Gambian D}32{,}051.28$$

5. How much in Leones will you get for the following amounts? Use the buying rate
   a. GH¢5,000.00 b. €200.00 c. ₦1,500.00

   Solution:

   a.
   $$\text{GH¢ } 1.00 = \text{Le } 1{,}500.00$$
   $$\text{GH¢ } 5{,}000 = 5{,}000 \times 1{,}500 = \text{Le } 7{,}500{,}000.00$$

   b.
   $$€1.00 = \text{Le } 8{,}600.00$$
   $$€200 = 200 \times 8{,}600 = \text{Le } 1{,}720{,}000.00$$

   c.
   $$₦1.00 = \text{Le } 20.00$$
   $$₦1{,}500 = 1{,}500 \times 20 = \text{Le } 30{,}000.00$$

Practice

1. A gas cooker depreciates at a rate of 15% per annum. If its value after 2 years is Le 614,125.00, what was its original price?
2. A computer costs Le 2,500,000.00. Its value depreciates by 20% the first year, 15% the second year and 12% the third year.
   a. What is its value at the end of the third year?
   b. If the owner decides to sell it at that price, what was the percentage loss on the original price to the nearest whole number?
3. Mr. Koroma and Mr. Kamara entered into a financial partnership with a total capital of Le 45,000,000.00. They agreed to contribute the capital in the ratio 2 : 1 respectively. The profit was shared as follows: Mr. Koroma was paid 6% of the total profit for his services as a manager. Each partner was paid 4% of the capital he invested. The remainder of the profit was then shared in the ratio of the capital invested. If Mr. Koroma's share of the total profits was Le 4,000,000.00, find:
   a. The total profit for the year to the nearest thousand Leones.
   b. Mr. Kamara's share of the total profits.
4. How much profit will a bank make if they buy, and then sell $500.00? Give your answer in Leones.
5. Mrs. Sesay buys goods from all over the world for her shop. She wants to order Le 10,000,000.00 worth of goods each from Nigeria, Great Britain and Germany. How much of each country's currency will she need?

Lesson Title: Indices	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L012	**Class:** SSS 4

Learning Outcomes

By the end of the lesson, you will be able to:

1. Apply the laws of indices to simplify expressions.
2. Solve equations that involve indices.

Overview

Numbers written with a base and a power are written in **index form**. For example, 7^3 is in index form, and it can be written $7^3 = 7 \times 7 \times 7$. Seven is the **base** and three is the **power** or **index**.

$$\text{base} \rightarrow 7^{3 \leftarrow \text{power/index}}$$

The following 4 **laws of indices** will be applied in this lesson:

1. To multiply two or more indices with the same base, add the powers: $a^m \times a^n = a^{m+n}$.
2. To divide two or more indices with the same base, subtract the powers: $a^m \div a^n = a^{m-n}$
3. If an index is raised to another power, multiply the indices: $(a^x)^y = a^{xy}$
4. Any non-zero number raised to the power zero is equal to 1: $a^0 = 1$

You also need to be familiar with **negative powers**. When indices have negative powers, the expression can be written as a fraction. The expression is moved to the denominator and the index is changed to a positive power. The numerator is 1. For example: $3^{-2} = \frac{1}{3^2}$ and $7^{-3} = \frac{1}{7^3}$. The general rule is written as: $a^{-n} = \frac{1}{a^n}$.

It is also important to be familiar with **fractional indices,** or indices where the power is a fraction. These are related to surds (for example, $\sqrt{4} = 4^{\frac{1}{2}}$). The general form is $\sqrt[n]{x} = x^{\frac{1}{n}}$. It is also possible that the numerator of the fraction is greater than 1. This means that there is a power on x within the surd. The general form is $\sqrt[n]{x^m} = x^{\frac{m}{n}}$.

Equations that involve indices are known as **exponential equations**. To solve exponential equations, you need to have indices with the same base on either side of the equals sign, so you can compare the powers and solve. For example, consider the equation: $10^{1-x} = 10^4$. The bases are the same (10), so the powers can be set equal to each other, and we can solve for x.

Solved Examples

1. Simplify: $\frac{3x^5y \times 4xy^3}{2x^4y^7}$

Solution:

$\frac{3x^5y\times 4xy^3}{2x^4y^3}$	$=$	$\frac{(3\times 4)x^{(5+1)}y^{1+3}}{2x^4y^3}$	Multiply coefficients and apply the first law of indices to the numerator.
	$=$	$\frac{12x^6y^4}{2x^4y^3}$	
	$=$	$6x^{6-4}y^{4-3}$	Divide 12 by 2 and apply the second Law of indices.
	$=$	$6x^2y$	

2. Simplify: $\frac{(2a^2b)^3\times 4\sqrt{a^4b^6}}{4a^5b\times 2a^6b^4}$

   Solution:

$\frac{(2a^2b)^3\times 4\sqrt{a^4b^6}}{4a^5b\times 2a^6b^4}$	$=$	$\frac{2^3a^{2\times 3}b^3\times 4(a^4b^6)^{\frac{1}{2}}}{(4\times 2)a^{5+6}b^{1+4}}$	Apply Law 1, Law 3 and note that $\sqrt[n]{a}=a^{\frac{1}{n}}$
	$=$	$\frac{(2^3\times 4)a^6b^3\times a^{4\times\frac{1}{2}}b^{6\times\frac{1}{2}}}{8a^{11}b^5}$	
	$=$	$\frac{32a^6b^3\times a^2b^3}{8a^{11}b^5}$	Simplify
	$=$	$4a^{6+2-11}b^{3+3-5}$	Apply Laws 1 and 2
	$=$	$4a^{-3}b$	Simplify
	$=$	$\frac{4b}{a^3}$	Note that $a^{-n}=\frac{1}{a^n}$

3. Simplify: $4^n\times 8^{2n}\div 4^{2n}$

   Solution:

$4^n\times 8^{2n}\div 4^{2n}$	$=$	$(2^2)^n\times(2^3)^{2n}\div(2^2)^{2n}$	Change the bases to 2
	$=$	$2^{2\times n}\times 2^{3\times 2n}\div 2^{2\times 2n}$	Apply Law 3
	$=$	$2^{2n}\times 2^{6n}\div 2^{4n}$	Simplify
	$=$	$2^{2n+6n-4n}$	Apply Laws 1 and 2
	$=$	2^{4n}	Simplify

4. Simplify: $\frac{a^{\frac{1}{4}}\times a^{-\frac{2}{3}}}{a^{\frac{1}{6}}}$

   Solution:

$\frac{a^{\frac{1}{4}}\times a^{-\frac{2}{3}}}{a^{\frac{1}{6}}}$	$=$	$a^{\frac{1}{4}+\left(-\frac{2}{3}\right)-\left(\frac{1}{6}\right)}$	Apply Laws 1 and 2
	$=$	$a^{\frac{1}{4}-\frac{2}{3}-\frac{1}{6}}$	Simplify
	$=$	$a^{-\frac{7}{12}}$	

$$= \frac{1}{a^{\left(\frac{7}{12}\right)}} \qquad \text{Note that } a^{-n} = \frac{1}{a^n}$$

5. Evaluate: $(32)^{\frac{3}{5}}$

Solution:

$$\begin{aligned} (32)^{\frac{3}{5}} &= (2^5)^{\frac{3}{5}} && \text{Write 32 in index form with 2 as base} \\ &= 2^{5\times\frac{3}{5}} && \text{Apply Law 3} \\ &= 2^3 && \text{Simplify} \\ &= 8 \end{aligned}$$

6. Evaluate: $\left(\frac{8}{27}\right)^{-\frac{2}{3}}$

Solution:

$$\begin{aligned} \left(\frac{8}{27}\right)^{-\frac{2}{3}} &= \left(\frac{2^3}{3^3}\right)^{-\frac{2}{3}} && \text{Write 8 and 27 in index form} \\ &= \frac{1}{\left(\frac{2^3}{3^3}\right)^{\frac{2}{3}}} && \text{Note } a^{-n} = \frac{1}{a^n} \\ &= \frac{1}{\left(\frac{2}{3}\right)^{3\times\frac{2}{3}}} && \text{Apply Law 3} \\ &= \frac{1}{\left(\frac{2}{3}\right)^{2}} && \text{Simplify} \\ &= \frac{1}{\left(\frac{4}{9}\right)} \\ &= 1 \times \frac{9}{4} && \text{To divide, multiply the numerator by the reciprocal of the fraction.} \\ &= \frac{9}{4} = 2\frac{1}{4} \end{aligned}$$

7. Find the value of $(47)^0 + \left(\frac{8}{125}\right)^{\frac{1}{3}}$

Solution:

$$\begin{aligned} (47)^0 + \left(\frac{8}{125}\right)^{\frac{1}{3}} &= 1 + \left(\frac{2^3}{5^3}\right)^{\frac{1}{3}} && \text{Apply Law 4 and write 8 and 125 in index form} \\ &= 1 + \frac{2}{5} && \text{Apply Law 3} \\ &= 1\frac{2}{5} \end{aligned}$$

8. If $7^{5x-3} = 49^{x+4}$, find x

Solution:

7^{5x-3}	$=$	49^{x+4}	
7^{5x-3}	$=$	$7^{2(x+4)}$	Write 49 in index form with 7 as base
7^{5x-3}	$=$	$7^{2(x+4)}$	
$5x-3$	$=$	$2(x+4)$	Since the bases are the same for both sides of the equation, the powers must be the same.
$5x-3$	$=$	$2x+8$	Solve for x.
$5x-2x$	$=$	$8+3$	
$3x$	$=$	11	
x	$=$	$\frac{11}{3}$	

Practice

1. Simplify: $\frac{20a^5b^7}{4a^2b}$
2. Simplify: $\frac{2x^4y^5\times(3xy)^2}{6x^3y^8}$
3. Simplify: $\frac{3^{n-1}\times27^{n+1}}{81^n}$
4. Evaluate: $(64)^{\frac{2}{3}}$
5. Evaluate:$\left(\frac{1}{8}\right)^{-\frac{2}{3}}$
6. Find the value of $121^0+\left(\frac{16}{625}\right)^{\frac{1}{4}}+\left(\frac{8}{27}\right)^{\frac{1}{3}}$
7. If $5^{2x-1}=125$, find x.
8. Given $4^{2x+1}=\frac{16^{x-2}}{2^x}$, find x.

Lesson Title: Logarithms – Part 1	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L013	**Class:** SSS 4

Learning Outcomes

By the end of the lesson, you will be able to:

1. Identify the relationship between logarithms and indices, and use it to solve logarithms.
2. Use logarithm tables to solve problems involving logarithms and antilogarithms.

Overview

Logarithms are related to indices. Logarithms have the form $\log_b y = x$, where b is the base. $\log_b y = x$ can be read "log to base b of y equals x". Each equation written as a logarithm has an equivalent equation written as an index, as shown:

$y = b^x$ ← power (↑ base)

$\log_b y = x$ ← power (↑ base)

Here is an example of how an index can be rewritten as a logarithm:

$$4^4 = 256 \quad \Leftrightarrow \quad \log_4 256 = 4$$

We can use this fact to solve logarithms. For example, consider $x = \log_3 9$. The logarithm is equal to an unknown value, x. We can use the relationship to indices to solve for x. Convert the equation to one with an index, then apply the rules for solving equations involving indices.

We can write the logarithm of each number as a decimal number. Logarithms of numbers can be found either using calculators or the logarithm table. The logarithm of a number is given in two parts, an integer (or whole number) before the decimal point, and a fractional part after the decimal point. The whole number part is called the **characteristic** and the decimal part is called the **mantissa**.

The **characteristic** can be found by expressing the number you are finding the logarithm of in standard form. The power on the 10 is the characteristic. For example, numbers in standard form are written as $a \times 10^n$, where n is the characteristic of the number. As a general note, if the number is greater than 1, then the characteristic is the number of digits before the decimal point, minus one. For example, the characteristic of 104.6 is 2, since the whole number part has 3 digits.

The **mantissa** is found using a logarithm table. In the table, look for each digit of the number you are taking the logarithm of. The first 2 digits are in the far-left column. If there is a 3rd digit, it will correspond to one of the large columns marked 1-9. If there is a 4th digit, find it in the "mean differences" columns, and add the number found there to the 4-digit number that you found from the first 3 digits.

We can find the logarithm of **decimal numbers less than 1** using a similar process. The characteristic can also be found by writing the decimal number in standard form. Note that the power will be negative in this case. When a decimal number should be expressed in its standard form $a \times 10^{-n}$, the characteristic of the logarithm is $\bar{n}$. For example, note that $0.0314 = 3.14 \times 10^{-2}$. The power gives the characteristic, $\bar{2}$. This is pronounced **"bar" 2**, not "minus" 2.

The mantissa is found using a logarithm table, in the same way as with number greater than 1. Write the mantissa after the characteristic.

Antilogarithms are the opposite of logarithms. They "undo" logarithms. They are called "antilog" for short. The antilog brings the number back to its ordinary form from the logarithm form. So, if $X = \log b$, then antilog $X = b$

We use tables of antilogarithms to solve antilog problems. Antilogarithms and logarithms each have their own table. We look for antilog in a table the same way as logarithms. Remember that a logarithm of numbers is made up of two parts: the characteristics and the mantissa. When finding the antilog, we look for the fractional part only (mantissa) in the antilog table. The characteristic (integer) tells us where to move the decimal point in the result. We should always add 1 to the characteristic and move the decimal point from left to right that number of spaces. Remember, when finding logarithm, we minus 1 from the integer part, so we do the opposite here in getting the antilog.

Solved Examples

1. Solve $x = \log_3 9$

   Solution:

   $x = \log_3 9$

   $9 = 3^x$ Change to index form

   $3^2 = 3^x$ Substitute $9 = 3^2$

   $2 = x$

2. Find the value of $\log_3 27$

   Solution:

   This logarithm is not equal to anything, but we can still write it in index form. Set it equal to x, then solve for x to get its value.

   $\log_3 27 = x$

   $27 = 3^x$ Change to index form

   $3^3 = 3^x$ Substitute $27 = 3^3$

   $3 = x$

3. Find the value of p if $\log_{10} 0.0001 = p$

Solution:

Let $\log_{10} 0.0001 = p$

$0.0001 = 10^p$	Change to index form
$\frac{1}{10000} = 10^p$	Convert decimal to fraction
$\frac{1}{10^4} = 10^p$	Substitute $10{,}000 = 10^4$
$10^{-4} = 10^p$	
$-4 = p$	

4. Find the value of $\log_3 \frac{1}{81}$.

Solution:

Let $\log_3 \frac{1}{81} = x$

$\frac{1}{81} = 3^x$	Change to index form
$\frac{1}{3^4} = 3^x$	Substitute $81 = 3^4$
$3^{-4} = 3^x$	
$-4 = x$	

5. Find $\log_{10} 76.83$

Solution:

The characteristic is 1, since we have two digits in the whole number part (7 and 6). Now look for the decimal part in the log table. Move along the row beginning with 76 and under 8, which gives 8,854. Next find the number in the difference column headed 3. The number is 2. Add the 2 to 8,854 to get 8,856.

Therefore $\log 76.83 = 1.8856$.

6. Find the logarithm of 432.5

Solution:

432.5	$= 4.325 \times 10^2$	Express in standard form
	$= 10^{0.6360} \times 10^2$	From the log table, look for row 43 and the column “2”, and difference 5 gives 6360
	$= 10^{0.6360+2}$	
	$= 10^{2.6360}$	

Hence $\log 432.5 = 2.6360$

7. Find the antilog of 0.5768

Solution:

Step 1. Go the antilog table. Look in the row marked .57, under the column headed by 6. The number here is 3,767.

Step 2. Look under column 8 in the “difference” section. The number there is 7.

Step 3. Add the numbers you got in steps 1 to 7, that is: $3767 + 7 = 3774$.

Step 4. The characteristic in the number 0.5768 is 0. Add 1 to this and move that number of spaces to the right. 3,774 □ 3.774

Answer: $\text{antilog}(0.5768) = 3.774$

8. Find the antilog of 3.7068.
   Solution:
   Step 1. Using the table, .706 □ 5,082
   Step 2. Difference from the '8' column: 9
   Step 3. Add: $5{,}082 + 9 = 5091$
   Step 4. Move 4 decimal places: 5,091 □ 5,091.0
   Answer: $\text{antilog}(3.7068) = 5091$

9. If $\log n = 2.3572$, find n.
   Solution:
   Take antilog of both sides: $\text{antilog}(\log n) = \text{antilog}(2.3572)$
   This gives: $n = \text{antilog}\ (2.3572)$
   Find the antilog of 2.3572: $n = 227.6$

10. Find the logarithms of the following numbers using the logarithm table:
    a. 0.153 b. 0.00654
    Solutions:

	Number	Logarithm	
a.	0.153	$\log 0.153$	$= \bar{1}.1847$
b.	0.00654	$\log 0.00654$	$= \bar{3}.8156$

11. Find the antilogarithm of the following:
    a. $\bar{1}.9178$ b. $\bar{2}.0451$
    Solution:

	Antilog	Value read from using mantissa antilog table	Number of zeros after the decimal	Answer
a.	$\bar{1}.9178$	8,275	0 zeros	0.8275
b.	$\bar{2}.0451$	1,109	1 zero	0.01109

Practice

1. Find the value of $\log_3 27$.
2. Find the value of $\log_2 0.25$.
3. Find the value of $\log_2 \sqrt{8}$.
4. Solve $\log_3 \frac{1}{243} = x$.
5. If $\log_2 x = 5$, find the value of x.
6. Use the logarithm table to find the logarithm of the following:
   a. 38 b. 4.643 c. 32.81
7. Find the antilog of the following:
   a. 1.2462 b. 3.1893 c. 2.8193
8. If $\log m = 2.1415$, find the value of m.

9. Find the logarithms of the following numbers:
    a. 0.123   b. 0.0987   c. 0.00654
10. Find the antilogarithm of the following:
    a. $\bar{2}.3067$ b. $\bar{4}.0101$ c. $\bar{1}.2011$

Lesson Title: Logarithms – Part 2	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L014	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to apply the laws of logarithms to solve problems.

Overview

The laws of logarithms are used to simplify expressions containing logarithms. The following are the laws of logarithms:

1. $\log_{10} pq = \log_{10} p + \log_{10} q$
2. $\log_{10}\left(\frac{p}{q}\right) = \log_{10} p - \log_{10} q$
3. $\log_{10}(P)^n = n\log_{10} P$

There are some other basic rules to keep in mind when simplifying logarithms. These are:

- The logarithm of any number to the base (non-zero number) of the same number is 1, that is $\log_a a = 1$.
- The logarithm of 1 to any base is 0, that is $\log_a 1 = 0$.

Solved Examples

1. Simplify the following: a. $\log 3 + \log 4$ b. $\log 3 + \log 2 + \log 5$
   Solutions:

   a. $\log 3 + \log 4 = \log(3 \times 4)$
   $= \log 12$

   b. $\log 3 + \log 2 + \log 5 = \log(3 \times 2 \times 5)$
   $= \log 30$

2. Given that $\log_{10} 3 = 0.4771$, $\log_{10} 4 = 0.6021$, $\log_{10} 5 = 0.6990$ and $\log_{10} 6 = 0.7782$, find the values of the following:
   a. $\log_{10} 9$ b. $\log_{10} 24$ e. $\log_{10} 72$
   Solutions:

   a. $\log_{10} 9 = \log_{10}(3 \times 3)$
   $= \log_{10} 3 + \log_{10} 3$
   $= 0.4771 + 0.4771$
   $= 0.9542$

   b. $\log_{10} 24 = \log_{10}(6 \times 4)$
   $= \log_{10} 6 + \log_{10} 4$
   $= 0.7782 + 0.6021$
   $= 1.3803$

   c. $\log_{10} 72 = \log_{10}(3 \times 4 \times 6)$
   $= \log_{10} 3 + \log_{10} 4 + \log_{10} 6$
   $= 0.4771 + 0.6021 + 0.7782$

$= 1.8574$

3. Simplify: a. $\log_3 21 - \log_3 7$ b. $\log_3 27 - \log_3 18$

Solutions:

a. $\log_3 21 - \log_3 7 = \log_3\left(\frac{21}{7}\right)$
$= \log_3 3$
$= 1$

b. $\log_3 27 - \log_3 18 = \log_3\left(\frac{27}{18}\right)$
$= \log_3\left(\frac{3}{2}\right)$
$= log_3 3 - log_3 2$
$= 1 - log_3 2$

4. Given that $\log_{10} 7 = 0.8451$, $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$, find the value of: a. $\log_{10}\left(\frac{9}{7}\right)$ b. $\log_{10} 0.21$

Solutions:

First, rewrite each mixed fraction as an improper fraction. Then, expand and solve by substituting the given values.

a. $\log_{10}\left(\frac{9}{7}\right)$
$= \log_{10} 9 - \log_{10} 7$
$= \log_{10} 3 \times 3 - \log_{10} 7$
$= \log_{10} 3 + \log_{10} 3 - \log_{10} 7$
$= 0.4771 + 0.4771 - 0.8451$
$=0.1091$

b. $\log_{10} 0.21 = \log \frac{21}{100}$
$= \log_{10} 21 - \log_{10} 100$
$= \log_{10}(7 \times 3) - \log_{10}(10 \times 10)$
$= \log_{10} 7 + \log_{10} 3 - (\log_{10} 10 + \log_{10} 10)$
$= 0.8451 + 0.4771 - (1 + 1)$
$= -0.6778$

5. Simplify the following expressions:

a. $\log_{10} 2^7$ b. $\log_5 a^3$ c. $\log_8 8^4$ d. $\log_2 4$

Solutions:

Rewrite each expression using the third law of logarithms:

a. $\log_{10} 2^7 = 7\log_{10} 2$
b. $\log_5 a^3 = 3\log_5 a$
c. $\log_8 8^4 = 4\log_8 8 = 4 \times 1 = 4$
d. For this example, first rewrite 4 as an index:
$\log_2 4 = \log_2 2^2$
Now apply the third law of logarithms:
$\log_2 2^2 = 2\log_2 2 = 2 \times 1 = 2$

6. Given $\log_{10} 3 = 0.4771$ and $\log_{10} 2 = 0.3010$ evaluate: a. $\log_{10} 32$ b. $\log_{10} 36$

Solutions:

a. $\log_{10} 32 = \log_{10} 2^5$
$= 5\log_{10} 2$
$= 5(0.3010)$
$= 1.505$

b. $\log_{10} 36 = \log_{10}(4 \times 9) = \log_{10} 4 + \log_{10} 9$

$$= \log_{10} 2^2 + \log_{10} 3^2$$
$$= 2\log_{10} 2 + 2\log_{10} 3$$
$$= 2(0.3010) + 2(0.4771) = 1.5562$$

7. Simplify:
   a. $\frac{\log_a 8 + \log_a 16 - \log_a 2}{\log_a 32}$
   b. $\frac{\log_2 8^3}{\log_2 8}$

   Solutions:

   a. $\frac{\log_a 8 + \log_a 16 - \log_a 2}{\log_a 32} = \frac{\log_a\left(\frac{8\times16}{2}\right)}{\log_a 32} = \frac{\log_a 64}{\log_a 32} = \frac{\log_a 2^6}{\log_a 2^5} = \frac{6(\log_a 2)}{5(\log_a 2)} = \frac{6}{5} = 1\frac{1}{5}$

   b. $\frac{\log_2 8^3}{\log_2 8} = \frac{3(\log_2 8)}{(\log_2 8)} = 3$

8. Simplify:
   a. $\log_{10} 25 + \log_{10} 4$
   b. $\log_3 15 + \log_3 9 - \log_3 5$

   Solutions:

   a.

$$\begin{aligned} \log_{10} 25 + \log_{10} 4 &= \log_{10}(25 \times 4) \\ &= \log_{10} 100 \\ &= \log_{10} 10^2 \\ &= 2\log_{10} 10 \\ &= 2 \times 1 = 2 \end{aligned}$$

   b.

$$\begin{aligned} \log_3 15 + \log_3 9 - \log_3 5 &= \log_3\left(\frac{15\times9}{5}\right) \\ &= \log_3 27 \\ &= \log_3 3^3 \\ &= 3(\log_3 3) \\ &= 3 \times 1 = 3 \end{aligned}$$

Practice

1. Simplify (write as single logarithm)
   a. $\log_2 6 + \log_2 8$
   b. $\log_5 4 + \log_5 2 + \log_5 3$
   c. $\log_4 8 + \log_4 8 + \log_4 2$
   d. $\log_3 9 + \log_3 2 + \log_3 2$

2. Given that $\log_{10} 2 = 0.3010$, $\log_{10} 5 = 0.6990$, and $\log_{10} 7 = 0.8451$, find the value of the following:
    a. $\log 28$
    b. $\log 56$
    c. $\log 70$

3. Simplify: a. $\log_4 32 - \log_4 8$ b. $\log_3 48 - \log_3 27$

4. Given that $\log_5 2 = 0.431$ and $\log_5 3 = 0.682$ find:
    a. $\log_5 6\frac{2}{3}$ b. $\log_5 0.6$ c. $\log_5 \left(\frac{9}{25}\right)$

5. Given that $\log_{10} 7 = 0.8451$, $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$ find:
    a. $\log_{10} \left(11\frac{4}{6}\right)$ b. $\log_{10} 6\frac{1}{8}$

6. Simplify: a. $\log_3 x^8$ b. $\log_2 8$ c. $\log_3 3^{17}$

7. Given $\log_5 2 = 0.431$ and $\log_5 3 = 0.682$ find the values of:
    a. $\log_5 8$
    b. $\log_5 \left(\frac{4}{5}\right)$
    c. $\log_5 18$

8. Simplify:
    a. $\frac{\log_{10} 625}{\log_{10} 25}$
    b. $\frac{\log_{10} a^4 - \log_{10} a^2}{\log_{10} a^3}$
    c. $\frac{\log_{10} 81}{\log_{10} 3}$

9. Evaluate: a. $3 + \log_3 81$ b. $(\log_5 125)^2 - \log_5 25$

10. Solve the simultaneous equations:
    $\log(a + b) = 0$
    $2loga = log(b + 1)$

Lesson Title: Representing sets with diagrams and symbols	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L015	**Class:** SSS 4

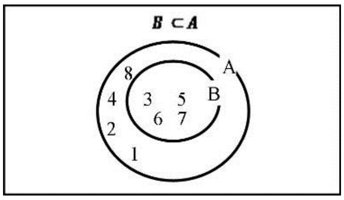
Learning Outcome
By the end of the lesson, you will be able to describe and represent sets using diagrams and symbols (including subsets, the intersection of 2 or 3 sets, disjoint sets, the union of 2 sets, the complement of a set).

Overview

A **universal set** is the set of all elements or objects under consideration. For example, consider a school. We have female and male students. The universal set is all pupils in the school. A universal set is denoted by $\mathcal{E}$ or U.

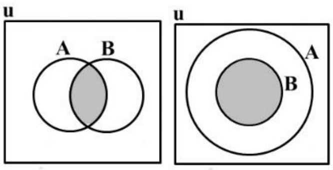
For any two sets A and B, if every element in set B is present in set A, then B is a **subset** of A. This is represented by $B \subset A$ and read as "set B is a subset of set A" or "B contained A". The notation $\not\subset$ means "not a subset of".

A **Venn diagram** is a diagram that shows the relationship between sets. A circle inside of another circle shows subsets. For example, this Venn diagram shows $B \subset A$, where $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{3, 5, 6, 7\}$.

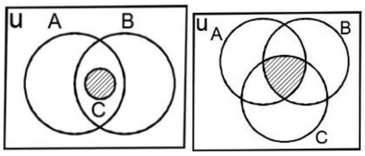


The **intersection** of 2 sets A and B is the element(s) that is/are common to both sets A and B. it is denoted by A ∩ B and is read as "A intersection B".

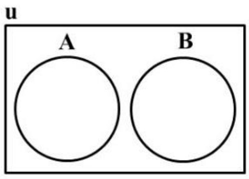
The intersection of two sets can be drawn using a Venn diagram. This is written A ∩ B, and read as "A intersection B". Possible intersections of A and B are shown by the shaded areas below. Note that in the second diagram, B ⊂ A and B ∩ A = B.



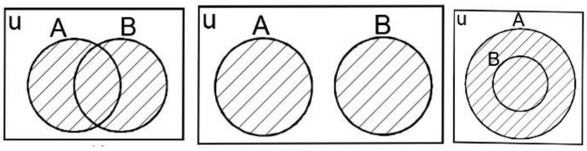
More than 2 sets can also intersect. The intersection of 3 sets A, B, and C is denoted A ∩ B ∩ C, and can be shown by the shaded areas below.



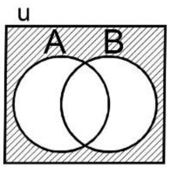
Any two sets are said to be **disjoint** sets if they have no element in common. That is to say their intersection is an empty/null set, $A \cap B = \emptyset$. This is shown in the Venn diagram on the right.



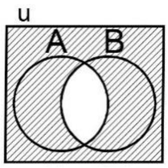
The **union** of any two sets say sets A and B is a set that contains elements or objects that belong to either sets A or B or to both. The union of two sets A and B is the set formed by putting the two sets together. The union of A and B is denoted $A \cup B$. It is read $'A$ union B'. If a member appears in both sets, it is listed only once in the union. The shaded portion of each diagram below shows the union of the two sets A and B.



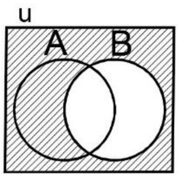
If U is the universal and A is a subset of U, then the **complement** of A is the set of members which belong to the universal set U but do not belong to A. In other words, the complement of a set is equal to elements in the universal set minus elements in the set. The complement of a set A is written as A' and is read as "A prime". The union of a set and its complement give the universal set ($A \cup A' = U$). The diagrams below illustrate complements of some sets.



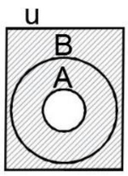
$(A \cup B)'$
"A union B, all prime"



$(A \cap B)'$
"A intersection B, all prime"



B'
"B prime"



$(A \subset B)'$
"prime of A subset B"

Solved Examples

1. Given the set M = {1, 2, 3, 4, 5, 6, 7, 8} which of the following sets are subsets of M?

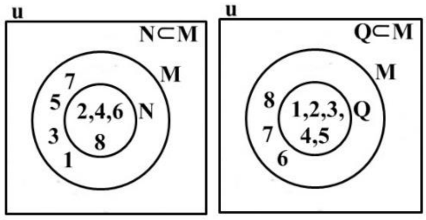
   a. i. N = {2, 4, 6, 8} ii. P = {1, 2, 5, 7, 9} iii. Q = {1, 2, 3, 4 5}

   b. Draw Venn diagrams to illustrate the relationships between set M and each of its subsets in part a.

   Solutions:

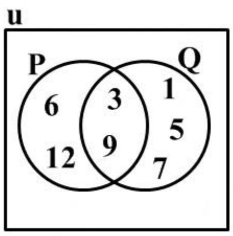
   a. i. N is a subset of M; All the elements in N are in M.
   ii. P is not a subset of M; 9 is an element of p but not in M.
   iii. Q is a subset of M; All the elements in Q are in M.

   b. Diagrams showing that N and Q are subsets of M:



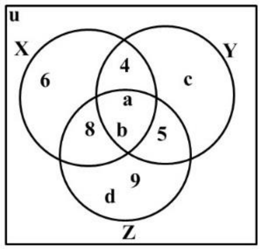
2. Consider two sets, P = {multiples of 3 less than 13} and Q = {Odd numbers less than 10}.
   a. List the members of both sets P and Q.
   b. Find P ∩ Q.
   c. Draw a Venn diagram to illustrate P ∩ Q.

   Solutions:
   a. P = {3, 6, 9, 12} and Q = {1, 3, 5, 7, 9}
   b. P ∩ Q = {3, 9}, because 3 and 9 are common to both sets.
   c. See the Venn diagram on the right.



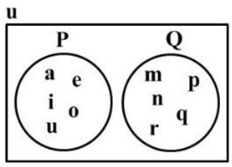
3. For the sets X = {4, 6, a, b, 8}, Y = {4, a, b, c, 5} and Z = {5, a, b, d, 8, 9},
   a. Find X∩Y, Y∩Z, X∩Z, and X∩Y∩Z.
   b. Illustrate your answer with a Venn diagram.

   Solutions:
   a. X ∩ Y = {4, a, b}; Y∩ Z = {5, a, b}; X∩Z = {8, a, b} and X∩Y∩Z = {a, b}
   b. See the Venn diagram on the right.



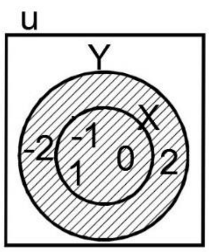
4. Given the sets, P = {a, e, i, o, u} and Q = {m, n, p, q, r} show that P and Q are disjoint sets and illustrate the two sets on a Venn diagram.

   Solution:
   P and Q share no common elements, so P∩Q = ∅. This is shown in the diagram on the right.



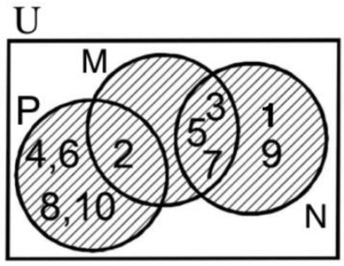
5. Given the set X = {x: $-2 < x < 2, x$ is an integer} and Y = {x: $-2 \leq x \leq 2, x$ is an integer}
   a. List the elements of X and Y.
   b. Find the union of X and Y.
   c. Show the union of X and Y on a Venn diagram.

   Solutions:
   a. X = {x: $-2 < x < 2$; x is an integer} = {−1, 0, 1}
      Y = {x: $-2 \leq x \leq 2, x$ is an integer} = {$-2, -1$, 0, 1, 2}
   b. X∪Y = {$-2,\ -1,\ 0,\ 1,\ 2,$} = Y, note that X⊂Y.
   c. See the Venn diagram on the right.



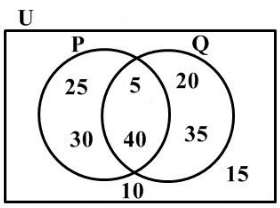
6. Given the sets M = $\{x: x$ is a prime number up to 10$\}$, N = $\{x: x$ is an odd positive integer up to 10$\}$ and P = $\{x: x$ is an even positive integer up to 10$\}$:
   a. List the elements of sets M, N and P.
   b. Find the union of sets M, N and P.
   c. Show the union of the sets on a Venn diagram.

   Solutions:
   a. M = {2, 3, 5, 7}, N = {3, 5, 7, 9}, P = {2, 4, 6, 8, 10}
   b. M∪N∪P = {2, 3, 4, 5, 6, 7, 8, 9, 10}
   b. See the Venn diagram on the right.



Practice

1. Represent the relationship between the following set B and its subsets V and C on a Venn diagram: B = $\{x: x$ is an alphabet between a and f$\}$, V = $\{x: x$ is a vowel$\}$ and C = $\{x: x$ is a consonant$\}$.
2. Consider two sets A and B, where A = {multiples of 4 less than 19} and B = {even numbers greater than 7 but less than 19}
   a. List the elements of set A and set B.
   b. Find A ∩ B.
   c. Illustrate set A and set B on a Venn diagram.
3. Consider the following sets, where U is the universal set: U = {2, 4, 6, 8...20}, A = {multiples of 4 less than 20}, B = {even numbers less than 14} and C = {8, 12, 16, 18, 20}.
   a. Find (i) A∩B, (ii) B∩C, (iii) A∩C, (iv) A∩B∩C.
   b. Draw a Venn diagram to illustrate A∩ B∩ C.
4. M, L, and T are subsets of the universal set U, where U = $\{x: 1 \leq x \leq 15\}$ and x is an integer. M consists of odd numbers, L consists of even numbers and T consists of multiples of 3.
   a. Find M∩L, M∩T, L∩T, and M∩L∩T.
   b. Draw a Venn diagram to illustrate U, M, L, and T.
5. Let U = {10, 20, 30, 40, 50, 60}. A and B are subsets of U such that A = {20, 30, 60}, B = {20, 30, 50}, find: a. A' b. B' c. A'∩B'
6. Using the Venn diagram, identify and list the following sets:
   a. U, P, and Q b. P' c. Q' d. P'∩Q'



Lesson Title: Solving problems involving sets	**Theme:** Numbers and Numeration
Practice Activity: PHM4-L016	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to diagram and solve real life problems involving 2 or 3 sets.

Overview

In this lesson, you will use your problem-solving skills and certain formulae (given below) to solve problems involving 2 or 3 sets.

The "cardinality of a set" or the "cardinal number of a set" is the number of elements in the set. It is denoted with the letter n and brackets. For example, consider $n(A) = 4$. This statement says that the set A has 4 elements.

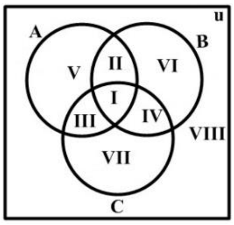
When we have two sets A and B then:

- $n(\mathrm{A} \cup \mathrm{B})$ is the number of elements present in either of the sets A or B.
- $n(\mathrm{A} \cap \mathrm{B})$ is the number of elements present in both the sets A and B.
- For two sets A and B: $n(\mathrm{A} \cup \mathrm{B}) = n(\mathrm{A}) + n(\mathrm{B}) - n(\mathrm{A} \cap \mathrm{B})$

This is one formula that is often used in problem solving with sets. The formula states that to find the total number of elements in A and B, then we add the number of elements in each set, and subtract the number of elements in their intersection. This is because we do not want to count the elements in their intersection twice.

If A, B and C are subsets of U the universal set then the Venn diagram below shows the various regions of the 3 sets. The following are sections of the diagram:

- I is the intersection of all the sets. $(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})$
- II is the elements of A and B only. $(\mathrm{C}' \cap (\mathrm{B} \cap \mathrm{A}))$
- III is the elements of A and C only. $(\mathrm{B}' \cap (\mathrm{A} \cap \mathrm{C}))$
- IV is the elements of B and C only. $(\mathrm{A}' \cap (\mathrm{B} \cap \mathrm{C}))$
- V is the elements in A only. $(\mathrm{A} \cap (\mathrm{B} \cup \mathrm{C})'$ or $\mathrm{A} \cap \mathrm{B}' \cap \mathrm{C}')$
- VI is the elements in B only. $(\mathrm{B} \cap (\mathrm{C} \cup \mathrm{A})'$ or $\mathrm{B} \cap \mathrm{C}' \cap \mathrm{A}')$
- VII is the elements in C only. $(\mathrm{C} \cap (\mathrm{B} \cup \mathrm{A})'$ or $\mathrm{C} \cap \mathrm{B}' \cap \mathrm{A}')$
- VIII is the elements that are not any of the 3 sets.



If 3 sets A, B and C are subsets of a universal set U, then the following rule holds:
$n(\mathrm{U}) = \mathrm{n}(\mathrm{A} \cup \mathrm{B} \cup \mathrm{C}) = \mathrm{n}(\mathrm{A}) + \mathrm{n}(\mathrm{B}) + \mathrm{n}(\mathrm{C}) - \mathrm{n}(\mathrm{A} \cap \mathrm{B}) - \mathrm{n}(\mathrm{A} \cap \mathrm{C}) - \mathrm{n}(\mathrm{B} \cap \mathrm{C}) + \mathrm{n}(\mathrm{A} \cap \mathrm{B} \cap \mathrm{C})$

Solved Examples

1. If A and B are two finite sets such that n(A) = 32, n(B) = 18 and n(A∪B) = 42, find n(A∩B).

   Solution:

   n(A∪B) = n(A) + n(B) – n(A∩B)

   n(A∩B) = n(A) + n(B) – n(A∪B)

   = 32+18 – 42

   = 50 – 42 = 8

2. A test was conducted for a class of 70 pupils. Fifty passed Mathematics and 40 passed English. Each student passed at least one subject.
   a. Illustrate this information on a Venn diagram.
   b. How many pupils passed both Mathematics and English?

   Solutions:

   Let U = {number of pupils that took the test}. Then, n{U} = 70

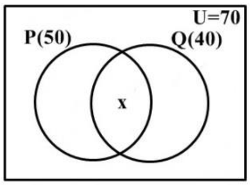
   P = {number of pupils who passed Mathematics}, and n{P} = 50

   Q = {number of pupils who passed English}, and n{Q} = 40

   x = number of pupils who passed both Maths and English = P∩Q

   n(x) = n(P∩Q)

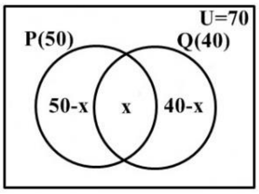
   a. The Venn diagram shows the above information. □

   Number of pupils who passed Mathematics only= $50 - x$

   Number of pupils who passed English only= $40 - x$

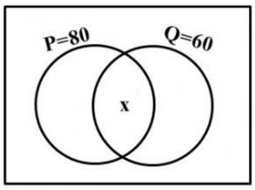
   This is shown in the Venn diagram. □

   b. To find x =n(P∩Q), add the number of elements in the three regions and equate it to n{U}. Solve the equation for x.

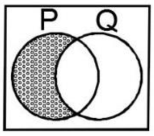
$$\begin{aligned}(50 - x) + x + (40 - x) &= 70\\ 50 - x + x + 40 - x &= 70\\ 50 + 40 - x &= 70\\ 90 - x &= 70\\ x &= 90 - 70 = 20\end{aligned}$$

3. In the Venn diagram on the right, set P has 80 members, set Q has 60 members while there are members in P∩Q and 120 members in P∪Q. Use the Venn diagram to answer the following questions:
   a. Shade the region which shows P only.
   b. Find the number of members in P∩Q.
   c. Find the number of members in Q only.

   Solutions:

a.



b. P∩Q $= x$

Region for P only $= 80 - x$

Region for Q only $= 60 - x$

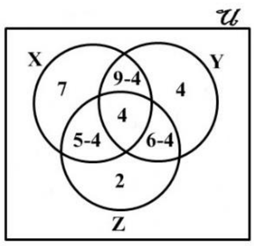
Add the three regions and equate to n(P∪Q)$= 120$

$$\begin{aligned} 80 - x + 60 - x + x &= 120 \\ 80 + 60 - x &= 120 \\ 140 - x &= 120 \\ x &= 140 - 120 \\ &= 20 \end{aligned}$$

c. Members in Q only $= 60 - x = 60 - 20 = 40$

4. Given three sets X, Y and Z is such that n(X∩Y∩Z) = 4, n(X∩Y) = 9, n(X∩Z) = 5, n(Y∩Z) = 6, n(X∩Y'∩Z') = 7, n(X'∩Y'∩Z) = 2 and n(X'∩Y∩Z') = 4,
   a. Draw the Venn diagram.
   b. Find: i. n(X) ii. n(Y) iii. n(Z)

Solutions:



a. See the Venn diagram on the right.

b. i. Find n(X) by adding all elements in set X:

n(X) = 7 + (9 – 4) + 4 + (5 – 4)

= 7 + 5 + 4 + 1 = 17

ii. Find n(Y) by adding all elements in set Y:

n(Y) = 4 + (9 – 4) + (6 – 4) + 4

= 4 + 5 + 2 + 4 = 15

iii. Find n(Z) by adding all elements in set Z:

n(Z) = 4 + (5 – 4) + 2 + (6 – 4)

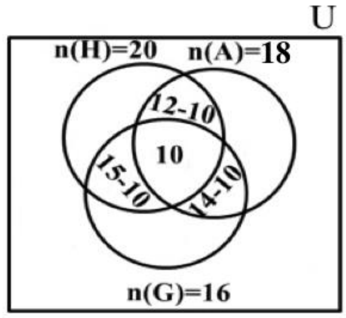
= 4 + 1 + 2 + 2 = 9

5. There are 73 students in a class. 20 of them study history, 18 study art and 16 study government. Ten study all three subjects, 15 study both history and government, 12 study both history and art, while 14 study both art and government.
   a. Illustrate the information on a Venn diagram.
   b. Using the Venn diagram, determine the number of students who study at least two subjects.

Solutions:

Let the set of students in the class be U; so total number of students will be n(U). Let the students studying each subject be given by H (history), A (art) and G (government). So n(H) = number of students who study history, n(A) = number of students who study art, and n(G) = number of students who study government. From the problem, we have n(U) = 73, n(H) = 20, n(A) = 18, n(G) = 16, n(H∩A∩G) = 10, n(H∩G) =15, n(H∩A) = 12 and n(A∩G) = 14.

a. See Venn diagram on the right.



b. The number of students that study at least two subjects will be represented by adding each segment in the intersections of the sets, as follows:

n(at least two subjects) = n(H∩A∩G') + n(H∩G∩A') + n(A∩G∩H') + n(H∩A∩G)

$= (15 - 10) + (14 - 10) + (12 - 10) + 10$

= 5 + 4 + 2 + 10

= 21

Practice

1. In a school of 120 students, 70 are on the Lawn tennis team and 80 are on the Volleyball team. Each student is in at least one team.
    a. Illustrate this information on a Venn diagram.
    b. Find how many students are on both teams.
    c. Find how many students are on the Lawn tennis team only.
2. In a village of 180 adult inhabitants, 60 speak only Mende fluently and 70 speak only Kono fluently. Each adult speaks at least one of the two languages.
    a. Draw a Venn diagram to illustrate this information.
    b. How many adults speak: i. Both languages fluently; ii. Kono; iii. Mende
3. In a class of 49 pupils, 27 offer French and 32 offer Krio. Nine pupils do not offer any of the languages.
    a. Draw a Venn diagram to illustrate this information.
    b. How many stunts offer: i. Both languages; ii. Only French; iii. Only Krio
4. In a class of 45 students, it is known that 24 study Music, 20 study Chemistry and 22 study Biology. All the students do at least one subject of the three. Three do all three subjects while 7 do Music and Biology. Six do Music and Chemistry but not Biology and 8 do Chemistry and Biology. How many students study: a. Only one subject; b. 2 subjects only
5. The Leone star coach invited 20 players for a soccer match. Eight of them were attackers, 12 of them were defenders and 12 were midfielders. The coach realised that 5 could play in the attack and in midfield, 6 defence and midfield, 2 only in attack and 3 play all the three roles.
    a. Illustrate the information on a Venn diagram.
    b. Find the number of players that can play:
    i. Only midfield; ii. Exactly two roles

Lesson Title: Operations on surds	**Theme:** Numbers and numeration
Practice Activity: PHM4-L017	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to perform operations on surds (addition, subtraction, multiplication).

Overview

Surds are numbers that we cannot find a whole number square root of, and are left in square root form to express their exact values. For example, the following are surds: $\sqrt{2}, \sqrt{3}, \sqrt{5}$. When calculated, surds have an infinite number of non-recurring decimals. Therefore, surds are irrational numbers. The square roots of all prime numbers are surds.

In simplifying surds to basic forms, it is often necessary to find the largest perfect square factor. You will use the fact that $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$. For example, consider $\sqrt{8}$. This is a surd that can be simplified. Note that 4 divides 8, and 4 is a perfect square.

Simplify $\sqrt{8}$ as follows: $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$.

In operating with surds note carefully the following properties of surds:

1. $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$
2. $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$
3. $\frac{b}{\sqrt{a}} = \frac{b}{\sqrt{a}} \times \frac{\sqrt{a}}{\sqrt{a}} = \frac{b\sqrt{a}}{a}$ (This is known as rationalising the denominator, and will be covered in the next lesson.)
4. $a\sqrt{c} \pm b\sqrt{c} = (a \pm b)\sqrt{c}$
5. $\sqrt{a^2} = \sqrt{a} \times \sqrt{a} = a$
6. $a\sqrt{b} \times \sqrt{c} = a\sqrt{bc}$
7. $a \times \sqrt{b} = a\sqrt{b}$
8. $a \times b\sqrt{c} = ab\sqrt{c}$

Addition and Subtraction

Surds should be added or subtracted in their basic (or simplified) form. **We can only add or subtract like surds. Like surds have the same surd part.** To add them, keep the same surd part, and add the coefficients. For example, $m\sqrt{k} + n\sqrt{k} = (m + n)\sqrt{k}$.

Mixed surds cannot be added or subtracted. That is, they cannot be simplified further. For example, $m\sqrt{k} + n\sqrt{p}$ or $m\sqrt{k} - n\sqrt{p}$ cannot be simplified further.

To add or subtract two or more surds, follow the following steps.

1. Convert each surd to its simplest form.
2. Add or subtract any like surds.

Multiplication

In the multiplication of surds, follow these steps carefully:

1. Express each surd in its basic or simplest form.
2. Multiply the coefficients, and multiply the surd parts
3. Make sure the result is in its simplest form, and simplify it if possible.

The general rules for multiplying surds are $\sqrt{a} \times \sqrt{\mathrm{b}} = \sqrt{ab}$ and $a\sqrt{b} \times c\sqrt{d} = ac\sqrt{bd}$.

Solved Examples

1. Simplify $\sqrt{147} - \sqrt{75} + \sqrt{27}$

   Solution:

$$\begin{aligned} \sqrt{147} - \sqrt{75} + \sqrt{27} &= \sqrt{49 \times 3} - \sqrt{25 \times 3} + \sqrt{9 \times 3} && \text{Simplify surds} \\ &= \sqrt{49} \times \sqrt{3} - \sqrt{25} \times \sqrt{3} + \sqrt{9} \times \sqrt{3} \\ &= 7\sqrt{3} - 5\sqrt{3} + 3\sqrt{3} \\ &= (7 - 5 + 3)\sqrt{3} && \text{Add/subtract} \\ &= 5\sqrt{3} \end{aligned}$$

2. Simplify $\frac{5}{2}\sqrt{32} - \frac{1}{5}\sqrt{125} - \frac{1}{3}\sqrt{72} + \frac{3}{2}\sqrt{180}$

   Solution:

$$\begin{aligned} \frac{5}{2}\sqrt{32} - \frac{1}{5}\sqrt{125} - \frac{1}{3}\sqrt{72} + \frac{3}{2}\sqrt{180} &= \frac{5}{2}\sqrt{16 \times 2} - \frac{1}{5}\sqrt{25 \times 5} - \frac{1}{3}\sqrt{36 \times 2} + \frac{3}{2}\sqrt{36 \times 5} \\ &= \frac{5}{2} \times 4\sqrt{2} - \frac{1}{5} \times 5\sqrt{5} - \frac{1}{3} \times 6\sqrt{2} + \frac{3}{2} \times 6\sqrt{5} \\ &= 10\sqrt{2} - \sqrt{5} - 2\sqrt{2} + 9\sqrt{5} \\ &= 10\sqrt{2} - 2\sqrt{2} + 9\sqrt{5} - \sqrt{5} \\ &= (10 - 2)\sqrt{2} + (9 - 1)\sqrt{5} \\ &= 8\sqrt{2} + 8\sqrt{5} \\ &= 8\left(\sqrt{2} + \sqrt{5}\right) \end{aligned}$$

3. If $\sqrt{294} + \sqrt{216} - \sqrt{486} = m\sqrt{6}$. Find the value of m.

   Solution:

   Simplify the left-hand side, then add and subtract the surds. Based on the right-hand side, it looks like we will have $\sqrt{6}$ in the surds on the left-hand side.

$$\sqrt{294} + \sqrt{216} - \sqrt{486} = \sqrt{49 \times 6} + \sqrt{36 \times 6} - \sqrt{81 \times 6}$$
$$= \sqrt{49} \times \sqrt{6} + \sqrt{36} \times \sqrt{6} - \sqrt{81} \times \sqrt{6}$$
$$= 7\sqrt{6} + 6\sqrt{6} - 9\sqrt{6}$$
$$= (7 + 6 - 9)\sqrt{6} = 4\sqrt{6}$$

Hence $m = 4$

4. Find the product of $7\sqrt{2}$ and $5\sqrt{3}$.

   Solution:

   $7\sqrt{2} \times 5\sqrt{3} = (7 \times 5)\sqrt{2 \times 3}$ Multiply coefficients and numbers in surds

   $= 35\sqrt{6}$

   The result is already in its simplest form, so the answer is $35\sqrt{6}$.

5. Simplify $\left(2\sqrt{3}\right)^2$

   Solution:

   $\left(2\sqrt{3}\right)^2 = 2\sqrt{3} \times 2\sqrt{3}$ Multiply

   $= (2 \times 2)\sqrt{3 \times 3}$ Multiply coefficients and numbers in surds

   $= 4\sqrt{9}$

   $= 4 \times 3$

   $= 12$

6. Simplify $\sqrt{10}(\sqrt{8} + 2)$

   Solution:

$$\sqrt{10}(\sqrt{8} + 2) = \sqrt{10} \times \sqrt{8} + 2 \times \sqrt{10}$$
$$= \sqrt{10 \times 8} + 2\sqrt{10}$$
$$= \sqrt{80} + 2\sqrt{10}$$
$$= \sqrt{16 \times 5} + 2\sqrt{10}$$
$$= \sqrt{16} \times \sqrt{5} + 2\sqrt{10}$$
$$= 4\sqrt{5} + 2\sqrt{10}$$

7. Simplify $\sqrt{48} - 3\sqrt{2}\left(2\sqrt{3} - 4\right) - 4\sqrt{48}$

   Solution:

$$\sqrt{48} - 3\sqrt{2}\left(2\sqrt{3} - 4\right) - 4\sqrt{48} = \sqrt{16 \times 3} - 3 \times 2\sqrt{2 \times 3} + 3 \times 4\sqrt{2} - 4\sqrt{16 \times 3}$$
$$= \sqrt{16} \times \sqrt{3} - 6\sqrt{6} + 12\sqrt{2} - 4 \times \sqrt{16} \times \sqrt{3}$$
$$= 4\sqrt{3} - 6\sqrt{6} + 12\sqrt{2} - 4 \times 4\sqrt{3}$$
$$= 4\sqrt{3} - 6\sqrt{6} - 12\sqrt{2} - 16\sqrt{3}$$
$$= -12\sqrt{2} - 12\sqrt{3} - 6\sqrt{6}$$

8. Simplify $(\sqrt{2} - 1)(3 - \sqrt{2})$ and write the result in the form $a + b\sqrt{c}$, stating the value of a, b and c.

Solution:

$$\begin{aligned}(\sqrt{2}-1)(3-\sqrt{2}) &= \sqrt{2}(3-\sqrt{2}) - 1(3-\sqrt{2}) \\ &= 3\sqrt{2} - \sqrt{2}\times\sqrt{2} - 3 + \sqrt{2} \\ &= 3\sqrt{2} - 2 - 3 + \sqrt{2} \\ &= -5 + 4\sqrt{2}\end{aligned}$$

Hence $a = -5, b = 4$ and $c = 2$

Practice

Simplify the following:

1. $3\sqrt{8} + \sqrt{50}$
2. $\frac{2}{3}\sqrt{63} + \frac{1}{2}\sqrt{28} - \frac{1}{10}\sqrt{175}$
3. $5\sqrt{125} - 2\sqrt{500} + 2\sqrt{20}$
4. $3\sqrt{7} - \sqrt{343} + 4\sqrt{28}$
5. If $\sqrt{175} - \sqrt{448} + \sqrt{112} = n\sqrt{7}$, find the value of n.
6. Find the product of:
   a. $7\sqrt{3}$ and $5\sqrt{3}$
   b. $\sqrt{32}$ and $\sqrt{80}$
7. Simplify:
   a. $4\sqrt{2} \times 2\sqrt{8}$
   b. $4\sqrt{12} \times 6\sqrt{18}$
8. Multiply:
   a. $\sqrt{64} \times \sqrt{48}$
   b. $2\sqrt{75} \times \sqrt{125}$
9. Simplify:
   a. $(3\sqrt{2})^2$
   b. $\sqrt{54} \times \sqrt{128}$
10. Simplify: $\sqrt{20} \times (\sqrt{5})^2$
11. Simplify: $\sqrt{2}(3 - 2\sqrt{2})$
12. Simplify: $\sqrt{18}\left(\sqrt{27} - \frac{2}{\sqrt{2}}\right)$
13. $2\sqrt{5}(6 - 3\sqrt{5})$
14. Simplify: $\sqrt{18}(3 + 2\sqrt{8}) + 3\sqrt{2}$ and write your result in the form $a + b\sqrt{c}$
15. Simplify: $\frac{\sqrt{80}\times\sqrt{63}}{\sqrt{125}\times\sqrt{28}}$
16. Simplify: $(2 + 3\sqrt{3})^2$ and write the result in the form $a + b\sqrt{c}$

Lesson Title: Simplifying surds	Theme: Numbers and numeration
Practice Activity: PHM4-L018	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Rationalise the denominator of surds.
2. Expand and simplify expressions involving surds.

Overview

Rationalising the Denominator

Rationalising the denominator of a surd is the process of changing the denominator from a surd to a rational number. The denominator of a fraction should not be an irrational number. To make the surd in the denominator a rational number, we multiply both the numerator and denominator by the surd.

The general rule is $\frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{b}$.

In some cases, there are 2 terms in the numerator or denominator, and at least one of them is a surd. For example, consider $\frac{1}{3-\sqrt{2}}$. We use a special process to rationalise the denominator of such fractions. Multiply the numerator and denominator by the **conjugate** of the binomial. To find the conjugate, change the sign in the middle. The conjugate of $3-\sqrt{2}$ is $3+\sqrt{2}$.

When a binomial is multiplied by its conjugate, simply square each term to save time. For example, $(3-\sqrt{2})(3+\sqrt{2}) = 3^2 - \sqrt{2}^2$. This follows the rule for finding the difference of 2 squares, $a^2 - b^2 = (a+b)(a-b)$.

You will also see problems with 2 terms in the numerator, such as $\frac{\sqrt{3}+\sqrt{7}}{\sqrt{21}}$. These problems are solved using the process above. Multiply the numerator and denominator by $\sqrt{21}$ to rationalise the denominator, then simplify.

Expanding and Simplifying

In expanding a bracket that involves surds you use the distributive law. For example, consider $\sqrt{5}\left(\sqrt{45} - \frac{8}{\sqrt{80}}\right)$. The term in front of the bracket ($\sqrt{5}$) will be multiplied by each term inside brackets.

You will also see problems that involve multiplication of 2 binomials, such as $(1+\sqrt{5})(1-\sqrt{5})$. Distribute as usual. Multiply each term in the first bracket by the expression in the second bracket.

Remember to follow the rules of multiplying surds. Rational numbers can be multiplied by each other, and surds can be multiplied by each other. However, numbers and surds behave like in algebra and cannot be combined (For example, $2 \times \sqrt{5} = 2\sqrt{5}$, and this cannot be simplified further).

Solved Examples

1. Simplify: $\frac{1}{\sqrt{2}}$

   Solution:

   $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ Multiply top and bottom by $\sqrt{2}$

   $= \frac{1\times\sqrt{2}}{\sqrt{2}\times\sqrt{2}}$

   $= \frac{\sqrt{2}}{2}$ Note : $\sqrt{2} \times \sqrt{2} = 2$

2. Simplify the following:

   a. $\frac{4}{\sqrt{24}}$ b. $\frac{2\sqrt{2}}{\sqrt{12}}$ c. $\frac{5}{3\sqrt{5}}$ d. $\frac{3\sqrt{3}}{2\sqrt{5}}$

   Solutions:

   a. $\frac{4}{\sqrt{24}} = \frac{4}{2\sqrt{6}}$

   $= \frac{2}{\sqrt{6}}$

   $= \frac{2}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$

   $= \frac{2\sqrt{6}}{6}$

   $= \frac{\sqrt{6}}{3}$

   b. $\frac{2\sqrt{2}}{\sqrt{12}} = \frac{2\sqrt{2}}{2\sqrt{3}}$

   $= \frac{\sqrt{2}}{\sqrt{3}}$

   $= \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

   $= \frac{\sqrt{6}}{3}$

   c. $\frac{5}{3\sqrt{5}} = \frac{5}{3\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

   $= \frac{5\sqrt{5}}{3\times5}$

   $= \frac{5\sqrt{5}}{15}$

   $= \frac{\sqrt{5}}{3}$

   d. $\frac{3\sqrt{3}}{2\sqrt{5}} = \frac{3\sqrt{3}}{2\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

   $= \frac{3\sqrt{15}}{2\times5}$

   $= \frac{3\sqrt{15}}{10}$

3. Simplify: $\frac{1}{3-\sqrt{2}}$

   Solution:

   Note that you will need to multiply the numerator and denominator by $3 + \sqrt{2}$, the conjugate of $3 - \sqrt{2}$.

   $\frac{1}{3-\sqrt{2}} = \frac{1}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}}$ Multiply top and bottom by $3 + \sqrt{2}$

   $= \frac{3+\sqrt{2}}{3^2-\sqrt{2}^2}$

   $= \frac{3+\sqrt{2}}{9-2}$ Simplify

   $= \frac{3+\sqrt{2}}{7}$

4. Simplify the following:

a. $\frac{2}{3+2\sqrt{2}}$ b. $\frac{5}{9-3\sqrt{5}}$

Solutions:

a. $\frac{2}{3+2\sqrt{2}}$
$$= \frac{2}{3+2\sqrt{2}} \times \frac{3-2\sqrt{2}}{3-2\sqrt{2}}$$
$$= \frac{2(3-2\sqrt{2})}{(3)^2-(2\sqrt{2})^2}$$
$$= \frac{6-4\sqrt{2}}{9-8}$$
$$= \frac{6-4\sqrt{2}}{1}$$
$$= 6-4\sqrt{2}$$

b. $\frac{5}{9-3\sqrt{5}}$
$$= \frac{5}{9-3\sqrt{5}} \times \frac{9+3\sqrt{5}}{9+3\sqrt{5}}$$
$$= \frac{5(9+3\sqrt{5})}{9^2-(3\sqrt{5})^2}$$
$$= \frac{45+15\sqrt{5}}{81-45}$$
$$= \frac{45+15\sqrt{5}}{36}$$
$$= \frac{45}{36} + \frac{15}{36}\sqrt{5}$$
$$= \frac{5}{4} + \frac{5}{12}\sqrt{5}$$

5. Express $\frac{\sqrt{3}+\sqrt{7}}{\sqrt{21}}$ in the form $a\sqrt{3} + b\sqrt{7}$, where a and b are rational numbers.

Solution:

$$\frac{\sqrt{3}+\sqrt{7}}{\sqrt{21}} = \frac{\sqrt{3}+\sqrt{7}}{\sqrt{21}} \times \frac{\sqrt{21}}{\sqrt{21}}$$ Rationalise the denominator
$$= \frac{\sqrt{21}(\sqrt{3}+\sqrt{7})}{\sqrt{21}\times\sqrt{21}}$$
$$= \frac{\sqrt{63}+\sqrt{147}}{21}$$ Multiply the numerator
$$= \frac{\sqrt{9\times 7}+\sqrt{49\times 3}}{21}$$ Simplify surds
$$= \frac{3\sqrt{7}+7\sqrt{3}}{21}$$
$$= \frac{7\sqrt{3}+3\sqrt{7}}{21}$$ Change the order of the numerator
$$= \frac{7\sqrt{3}}{21} + \frac{3\sqrt{7}}{21}$$ Simplify
$$= \frac{1}{3}\sqrt{3} + \frac{1}{7}\sqrt{7}$$ where $a = \frac{1}{3}$ and $b = \frac{1}{7}$

6. Simplify $\frac{1+\sqrt{2}}{\sqrt{2}}$ to the form $m + n\sqrt{2}$.

Solution:

$$\frac{1+\sqrt{2}}{\sqrt{2}} = \frac{1+\sqrt{2}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\sqrt{2}(1+\sqrt{2})}{\sqrt{2}\times\sqrt{2}}$$
$$= \frac{\sqrt{2}+2}{2}$$
$$= \frac{\sqrt{2}}{2} + \frac{2}{2}$$
$$= 1 + \frac{1}{2}\sqrt{2}$$ where $m = 1$ and $n = \frac{1}{2}$

7. Expand and simplify $(1+\sqrt{5})(1-\sqrt{5})$

Solution:

$(1+\sqrt{5})(1-\sqrt{5})$	$=$	$1(1-\sqrt{5})+\sqrt{5}(1-\sqrt{5})$	Expand (distribute)
	$=$	$1-\sqrt{5}+\sqrt{5}-(\sqrt{5})^2$	Simplify
	$=$	$1-\sqrt{5}+\sqrt{5}-5$	Note $(\sqrt{5})^2=\sqrt{5}\times\sqrt{5}=5$
	$=$	$1-5-\sqrt{5}+\sqrt{5}$	Collect like terms
	$=$	-4	

8. Expand and simplify $\sqrt{5}\left(\sqrt{45}-\frac{8}{\sqrt{80}}\right)$.

Solution:

$\sqrt{5}\left(\sqrt{45}-\frac{8}{\sqrt{80}}\right)$	$=$	$\sqrt{5\times 45}-\sqrt{5}\times\frac{8}{\sqrt{80}}$	Multiply $\sqrt{5}$ by each term in the bracket
	$=$	$\sqrt{225}-\frac{8\sqrt{5}}{\sqrt{80}}$	
	$=$	$15-\frac{8\sqrt{5}}{\sqrt{16\times 5}}$	
	$=$	$15-\frac{8\sqrt{5}}{4\sqrt{5}}$	
	$=$	$15-\frac{8}{4}$	Cancel $\sqrt{5}$
	$=$	$15-2$	
	$=$	13	

Practice

1. Simplify: a. $\frac{3}{\sqrt{8}}$ b. $\frac{16}{\sqrt{128}}$ c. $\frac{\sqrt{27}}{\sqrt{72}}$ d. $\frac{7\sqrt{2}}{\sqrt{14}}$ e. $\frac{4\sqrt{3}}{3\sqrt{2}}$
2. Simplify:
   a. $\frac{3}{4-\sqrt{3}}$ b. $\frac{1}{3+\sqrt{2}}$ c. $\frac{5}{5+\sqrt{2}}$ d. $\frac{6}{7-2\sqrt{2}}$ e. $\frac{7}{8-2\sqrt{3}}$
3. Simplify the following:
   a. $\frac{9-\sqrt{108}}{\sqrt{72}}$ b. $\frac{\sqrt{5}+\sqrt{75}}{\sqrt{27}}$ c. $\frac{8+\sqrt{12}}{2\sqrt{5}}$
4. Simplify: $\sqrt{2}(\sqrt{10}+\sqrt{2})$
5. Expand and simplify: $\sqrt{18}(2\sqrt{5}-\sqrt{3})$
6. Expand and simplify:
   a. $(1+\sqrt{2})(1-\sqrt{2})$
   b. $(2-3\sqrt{2})(3\sqrt{2}+1)$
7. Expand and simplify: $(1+\sqrt{2})^2$
8. Express $\frac{4+5\sqrt{2}}{1-2\sqrt{2}}$ in the form: $a+b\sqrt{2}$

Lesson Title: Simplification and factorisation	Theme: Algebra
Practice Activity: PHM4-L019	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to simplify and factor algebraic expressions.

Overview

Simplifying algebraic expressions

Algebraic expressions can be simplified by combining like terms and removing brackets.

The first step in simplifying an expression is to **remove brackets**, if there are any. In removing brackets, multiply the term outside the bracket by each of the terms inside the bracket. For example, in the expression $3(x+2)$, each term inside the brackets is multiplied by 3. Thus, $3(x+2)=3x+6$.

In some cases, there are 2 sets of brackets, for example $(x+5)(x+10)$. In this expression, 2 binomials are multiplied. Binomials are algebraic expressions with 2 terms. Multiply each term in one bracket by each term in the other bracket. For example, $(x+5)(x+10)=x(x+10)+5(x+10)$. Each set of brackets must now be removed: $x(x+10)+5(x+10)=x^2+10x+5x+50$.

We must be very careful with signs when removing brackets. When there is a negative sign (–) in front of the brackets, each sign inside the bracket changes when the brackets are removed. This is because of the rules of multiplication. Remember that multiplying a negative by a positive gives a negative. Multiplying a negative and a negative gives a positive.

After removing brackets in an expression, collect any like terms together and combine them.

Like terms are terms that have the same variable, and the variables have the same power. For example, $2a, 3a, 5a$ are all like terms, with the variable a to the power 1. As another example, $5p^2$ and $8p^2$ are like terms.

Like terms can be combined by adding or subtracting to give a single term. The result will have the same variable with the same power. The coefficients of the terms are added or subtracted. We can further simplify the example given above by combining the like terms $10x$ and $5x$. That is, $x^2+10x+5x+50=x^2+(10+5)x+50=x^2+15x+50$.

This is the most simplified form of our example. We now have $(x+5)(x+10)=x^2+15x+50$.

Factorising algebraic expressions

Factorising in algebra is the reverse of simplifying (or expanding). Consider $2(a+b) = 2a + 2b$. You can simplify $2(a+b)$ by removing brackets.

If you switch sides in the above equation you will get $2a + 2b = 2(a+b)$. We can factorise $2a + 2b$ by taking the common factor 2 out of each term in $2a + 2b$ to get 2 multiplied by $(a+b)$.

The general method used to factorise terms with common factors in algebraic expressions is to find the greatest common factor (GCF) to all terms and bring it outside the bracket.

In more complicated algebraic expressions, you may need to group terms with common factors. For example, consider the algebraic expression: $x^2 - ax + bx - ab$

Notice that the 4 terms do not have 1 common factor. The absence of a common factor to all four terms does not mean the expression cannot be factorised. There are common factors to some pairs of terms which can be grouped for factorisation. For the example expression, we can form a group with a common factor of x, and another group with a common factor of b. We then factor x and b.

Form the two groups: $x^2 - ax + bx - ab = (x^2 - ax) + (bx - ab)$

Factor x and b: $x(x-a) + b(x-a)$

We now have 2 terms in the expression, $x(x-a)$ and $b(x-a)$. Notice that $(x-a)$ is a common factor in these terms. It can be factored out of the expression, leaving $(x+b)$:

$$x^2 - ax + bx - ab = x(x-a) + b(x-a) = (x-a)(x+b)$$

Solved Examples

1. Simplify: $12e + 5f - 4e - 2f$
   Solution:

$12e + 5f - 4e - 2f$	$=$	$12e - 4e + 5f - 2f$	Collect like terms
	$=$	$(12-4)e + (5-2)f$	Combine like terms
	$=$	$8e + 3f$	

2. Simplify: $8p - 3(2q + 2p)$
   Solution:

$8p - 3(2q + 2p)$	$= 8p + (-3)(2q) + (-3)(2p)$	Remove the brackets
	$= 8p - 6q - 6p$	
	$= 8p - 6p - 6q$	Collect like terms
	$= 2p - 6q$	

3. Remove brackets and simplify: $3(y+5)+6(y+2)$

Solution:

$3(y+5)+6(y+2)$ $= 3y+15+6y+12$ Remove the brackets

$= 3y+6y+15+12$ Collect like terms

$= 9y+27$

4. Remove brackets and simplify: $4(b-2a)+3(a-3b)-4(b-a)$

Solution:

$4(b-2a)+3(a-3b)-4(b-a)$ $= 4b-8a+3a-9b-4b+4a$

$= -8a+3a+4a+4b-9b-4b$

$= -a-9b$

5. Expand: $(y-4)(y-3)$

Solution:

$(y-4)(y-3)$ $= y(y-3)-4(y-3)$ Multiply

$= y^2-3y-4y+12$ Remove the brackets

$= y^2-7y+12$ Combine like terms

6. Expand and simplify: $(5m+6n)(u-v)+(3m+2n)(2u+v)$

Solution:

Step 1. Expand the first two brackets:

$(5m+6n)(u-v)$ $= u(5m+6n)-v(5m+6n)$ Multiply

$= 5mu+6nu-5mv-6nv$ Remove brackets

Step 2. Expand the last two brackets:

$(3m+2n)(2u+v)$ $= 2u(3m+2n)+v(3m+2n)$ Multiply

$= 6mu+4nu+3mv+2nv$ Remove brackets

Step 3. Collect like terms after the two expansions:

$5mu+6mu+6nu+4nu-5mv+3mv-6nv+2n$

$= 11mu+10nu-2mv-4nv$

7. Factorise:
   a. $64x^3+16x^2$
   b. $5a^4-10a^3+15ab$
   c. $3xy^3-3xy^2+6x^2y^2$

   Solutions:

   a. $16x^2$ is a common factor: $64x^3+16x^2 = 16x^2(4x+1)$
   b. $5a$ is a common factor: $5a^4-10a^3+15ab = 5a(a^3-2a^2+3b)$
   c. $3xy^2$ is a common factor: $3xy^3-3xy^2+6x^2y^2 = 3xy^2(y+2x-1)$

8. Factorise completely: $2a^3+4a^2-3a-6$

Solution:

$2a^3 + 4a^2 - 3a - 6$	$= (2a^3 + 4a^2) + (-3a - 6)$	Create smaller groups
	$= 2a^2(a + 2) - 3(a + 2)$	Factor out each GCF
	$= (a + 2)(2a^2 - 3)$	Factor out $(a + 2)$

9. Factorise completely: $2x^3 - 6x^2 + 10x - 30$

   Solution:

   In this case, all 4 terms have a common factor, 2. When there are common factors, we should first factor them out. We can then proceed with the steps described above on the 4 terms inside brackets.

$2x^3 - 6x^2 + 10x - 30$	$= 2(x^3 - 3x^2 + 5x - 15)$	Factor out 2
	$= 2[(x^3 - 3x^2) + (5x - 15)]$	Create smaller groups
	$= 2[x^2(x - 3) + 5(x - 3)]$	Factor out each GCF
	$= 2(x - 3)(x^2 + 5)$	Factor out $(x - 3)$

Practice

Simplify the following expressions:

1. $4y - 3x + 5x - 3y$
2. $12p^2q - 4pq^2 + pq^2 - 4p^2q$

Expand and simplify:

3. $11u - 3u(2v + 3)$
4. $8(-3m + 2n) - 2(m + n)$
5. $(m + 3)(m - 2)$
6. $(x - 4)^2$
7. $(n + 3)^2$
8. $(3m + 2)(n + 3) + (5m - 4)(n + 2)$
9. $3x(x - 2)^2$

Factorise the following expressions:

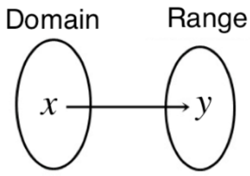
10. $9ab - 27$
11. $a^3 + ab$
12. $16x^4 - 8x^3 + 4x^2$
13. $81a^3 + 9a^2$
14. $18s^5t^6 - 6s^4t^3 + 3s^2t^2$
15. $5uv^3 - 25uv^2 + 15u^2v^2$
16. $cd - ce - d^2 + de$
17. $p + q + 5ap + 5aq$
18. $x^2 + 2x + 3x + 6$
19. $10x^2 - 11x - 6$
20. $2 - x - 6x^2$

Lesson Title: Functions	**Theme:** Algebra
Practice Activity: PHM4-L020	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Identify and describe functions, and their domain and range.
2. Use function notation.

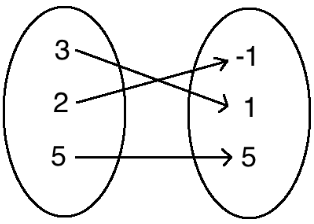
Overview

A function is a relation between inputs and outputs. The domain is the set of elements that can be put into the function, and the range is the set of outputs that the function produces. In algebra, we typically represent inputs with x and outputs with y.

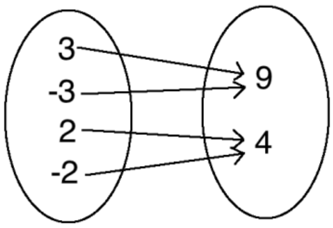


A function maps each element in the domain onto one and only one member of the range.

Functions are represented by the diagrams below. In each diagram, each member of the domain is mapped onto exactly one member of the range.

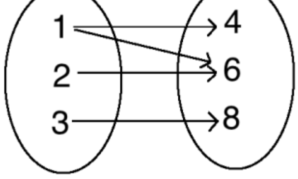


Function



Function

Note that the following diagram represents a relation that is **not** a function. A member of the first set (1) maps onto two members of the second set (4 and 6):



NOT a function

Function notation is another way of writing equations. You have probably seen equations written with y. In function notation, y is replaced with $f(x)$ (read as "f of x"). In other words, $y = f(x)$. $f(x)$ is used to show functions of x. In other words, the function is in terms of x.

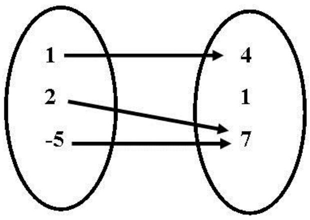
For example, $f(x) = 2x + 1$ is the same as $y = 2x + 1$. These are equivalent, and the second one is called a "function" as well. The first one is written in function notation.

The notation $f: x \longrightarrow y$ tells us that the function's name is "f" and its ordered pairs are formed by elements x from the domain and elements y from the range. The arrow $\longrightarrow$ is read "is mapped to".

The corresponding x- and y-values that satisfy functions can be written as ordered pairs (x, y). These are used in graphing functions on the Cartesian plane. This is covered in lessons on linear, quadratic, and trigonometric functions.

Solved Examples

1. Are the following functions?
    a. $\{(2,-1),(5,1),(-5,1)\}$
    b. $\{(6,3),(6,5)\}$
    c.



Solutions:

a. Yes; each member of the domain (2, 5, -5) maps to exactly 1 member of the co-domain;
b. No; a member of the domain (6) maps to 2 members of the co-domain;
c. Yes; each member of the domain (1, 2, -5) maps to exactly 1 member of the co-domain. It is okay if 2 members of the domain map to the same member of the co-domain.

2. If $f(x) = 3x + 4$, find $f(3)$ and $f(0)$
   Solution:
   This question asks us to find "f of 3" and "f of 0". This means that we should substitute $x = 3$ and $x = 0$ into the function and evaluate.

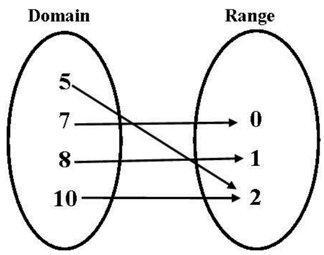
$f(x)$	$=$	$3x + 4$	Function
$f(3)$	$=$	$3(3) + 4$	Substitute $x = 3$
	$=$	$9 + 4$	Simplify
	$=$	13	
$f(0)$	$=$	$3(0) + 4$	Substitute $x = 0$
	$=$	$0 + 4$	Simplify
	$=$	4	

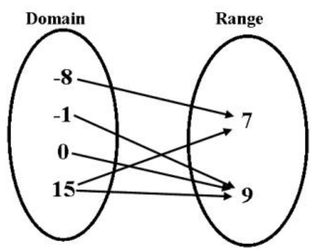
Answer: $f(3) = 13$ and $f(0) = 4$

3. A function is defined by $f: x \longrightarrow 4x + 1$ on the domain $\{-1, 0, 1, 2\}$. Find the range of the function.
   Solution:
   For this problem, we need to substitute each value from the domain to find the corresponding value in the range.
   $$f(-1) = 4(-1) + 1 = -4 + 1 = -3$$
   $$f(0) = 4(0) + 1 = 0 + 1 = 1$$
   $$f(1) = 4(1) + 1 = 4 + 1 = 5$$
   $$f(2) = 4(2) + 1 = 8 + 1 = 9$$
   Therefore, range = $\{-3, 1, 5, 9\}$.

4. A function $f: x \longrightarrow x^2 + 5$, is defined on the domain $\{0, 1, 2, 3\}$. Find the range.
   Solution:
   $$f(0) = (0)^2 + 5 = 0 + 5 = 5$$
   $$f(1) = (1)^2 + 5 = 1 + 5 = 6$$
   $$f(2) = (2)^2 + 5 = 4 + 5 = 9$$
   $$f(3) = (3)^2 + 5 = 9 + 5 = 14$$
   Therefore, the range is $\{5, 6, 9, 14\}$.

Practice

1. Determine whether the following relations are functions, and state your reason why.
   a. $\{(-3,7), (-1,5), (0,-2), (5,9), (5,3)\}$
   b. $\{(-2,0), (-1,-2), (0,3), (4,-1), (5,-3)\}$
2. Are the relations expressed in the mapping diagrams functions? Give your reasons.
   a.
   

   b.
   

3. If $f(x) = 2x + 3$, find: a. $f(1)$ b. $f(2)$ c. $f(-3)$
4. If $f(x) = x^2 + 2x - 3$, find: a. $f(0)$ b. $f(1)$ c. $f(-1)$
5. A function $f: x \longrightarrow 3x - 2$ is defined on the domain $\{4, 6, 8\}$. Find the range.
6. A function $f: x \longrightarrow x^2 + 1$ is defined on the domain $\{1, 2, 3\}$. Find the range.

Lesson Title: Graphing linear functions	**Theme:** Algebraic Processes
Practice Activity: PHM4-L021	**Class:** SSS 4

Learning Outcome

By the end of the lesson, you will be able to graph linear functions, and identify the solutions and gradient.

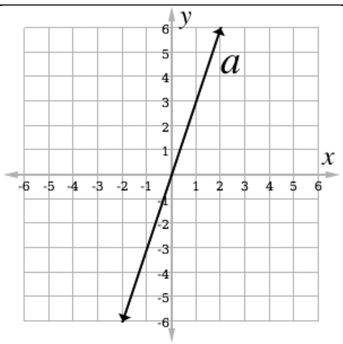
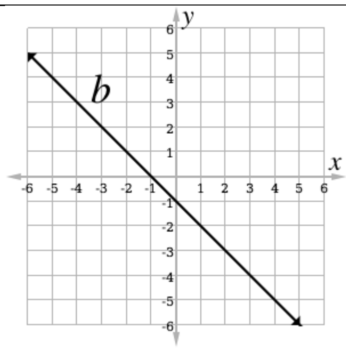
Overview

Solutions to linear equations can be written as ordered pairs: (x, y). To find a solution to a linear equation, substitute any value of x and solve for y.

To graph a linear equation, fill a table of values with multiple solutions to the equation. Graph each solution on the Cartesian plane, and connect them with a straight line.

Gradient is a number that tells us in which direction a line increases, and how steep it is. If a line increases as it goes to the right, or in the positive x-direction, the gradient is positive. If a line increases as it goes to the left, or in the negative x-direction, the gradient is negative.

The greater the absolute value of a gradient, the steeper the line is. Examples:

	
• Line a increases as it goes to the **right**, or the positive x-direction. Line a has a **positive** gradient. • Line a is steeper than line b. It has a gradient of +3.	• Line b increases as it goes to the **left**, or the negative x-direction. Line b has a **negative** gradient. • Line b is not as steep as line a. It has a gradient of -1.

The gradient of a line can be calculated using any 2 points on the line. It is calculated by dividing the change in y by the change in x between those 2 points.

$$gradient = \frac{rise}{run} = \frac{change\ in\ y}{change\ in\ x}$$

Gradient m is given by the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$ for any 2 points (x_1, y_1) and (x_2, y_2) on a line.

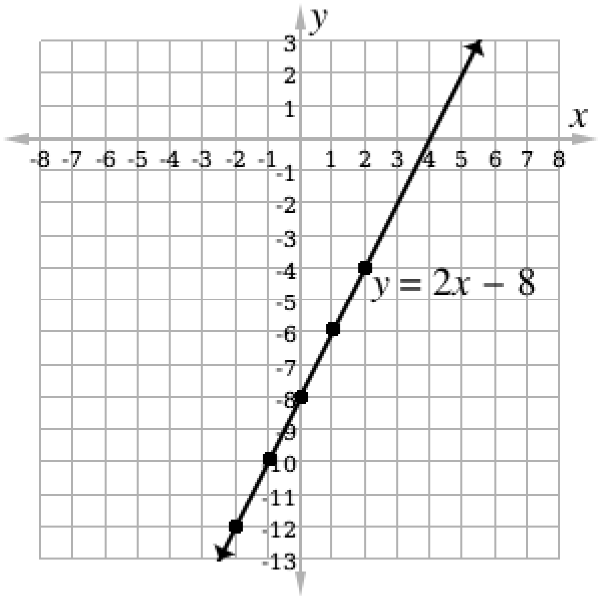
Solved Examples

1. Graph the equation $y = 2x - 8$

$$\begin{aligned} y &= 2(-2) - 8 \\ &= -4 - 8 \\ &= -12 \end{aligned}$$

x	-2	-1	0	1	2
y	-12	-10	-8	-6	-4

$$\begin{aligned} y &= 2(-1) - 8 \\ &= -2 - 8 \\ &= -10 \end{aligned}$$



$$\begin{aligned} y &= 2(0) - 8 \\ &= 0 - 8 \\ &= -8 \end{aligned}$$

$$\begin{aligned} y &= 2(1) - 8 \\ &= 2 - 8 \\ &= -6 \end{aligned}$$

$$\begin{aligned} y &= 2(2) - 8 \\ &= 4 - 8 \\ &= -4 \end{aligned}$$

2. Graph the equation $2y + 3x - 6 = 0$

Step 1. Make y the subject of the equation:

$$\begin{aligned} 2y + 3x - 6 &= 0 \\ 2y &= -3x + 6 && \text{Transpose } 3x \text{ and } -6 \\ y &= -\tfrac{3}{2}x + 3 && \text{Divide throughout by 2} \end{aligned}$$

Step 2. Fill the table of values and graph the line:

$$\begin{aligned} y &= -\tfrac{3}{2}(-2) + 3 \\ &= 3 + 3 \\ &= 6 \end{aligned}$$

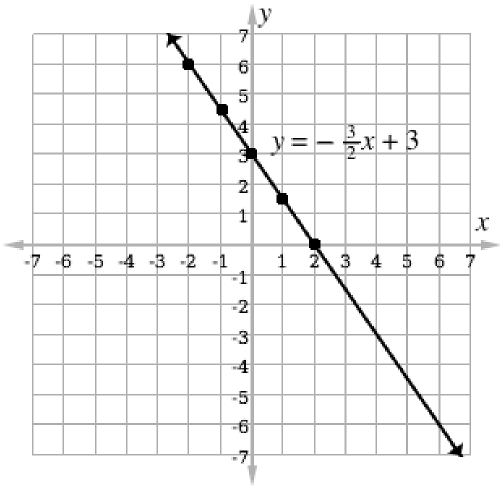
x	-2	-1	0	1	2
y	6	4.5	3	1.5	0

$$\begin{aligned} y &= -\tfrac{3}{2}(-1) + 3 \\ &= \tfrac{3}{2} + 3 \\ &= 4.5 \end{aligned}$$

$$
\begin{aligned}
y &= -\tfrac{3}{2}(0) + 3 \\
&= 0 + 3 \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
y &= -\tfrac{3}{2}(1) + 3 \\
&= -1.5 + 3 \\
&= 1.5
\end{aligned}
$$

$$
\begin{aligned}
y &= -\tfrac{3}{2}(2) + 3 \\
&= -3 + 3 \\
&= 0
\end{aligned}
$$



3. Find the gradient of the line passing through $(7, -12)$ and $(11, 4)$.
   Solution:

$$
\begin{aligned}
m &= \frac{4-(-12)}{11-7} && \text{Substitute } x\text{- and } y\text{-values} \\
&= \frac{16}{4} && \text{Simplify} \\
&= 4
\end{aligned}
$$

4. The gradient of the line joining the points $(p, 3)$ and $(4,8)$ is 5. Find p.
   Solution:

$$
\begin{aligned}
5 &= \frac{8-3}{4-p} && \text{Substitute in the gradient formula} \\
5(4-p) &= 5 && \text{Solve for } p \\
20 - 5p &= 5 \\
\frac{-5p}{-5} &= \frac{-15}{-5} && \text{Simplify} \\
p &= 3
\end{aligned}
$$

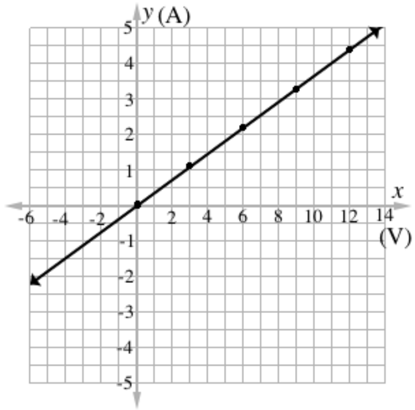
5. The current in amps in a wire for a given potential difference in volts is shown in the table.

p.d. (V)	0	3	6	9	12
Current (A)	0	1.2	2.4	3.6	4.8

   a. Draw a graph of this relationship, using a scale of 1 cm to 1 V on the horizontal axis and 2 cm to 1 A on the vertical axis.
   b. Find the current for a p.d. of: i. 1 V ii. 4 V
   c. Find the p.d. for a current of: i. 0.8 A ii. 2 A

Solutions:
This problem seems difficult because it is about a topic that many people do not understand well. However, you must simply plot the points given in the table. To solve parts b. and c., either use the graph or create a linear equation. Your graph must be drawn very accurately in order to solve by graphing.

a. Graph (not to scale):



b. Find the given values on the x-axis, and identify the corresponding y-value on the line. (Answers: i. 0.4 A; ii. 1.6 A)

Alternatively, find the slope and create a linear equation:

$$m = \frac{1.2-0}{3-0} \quad \text{Substitute } x\text{- and } y\text{-values}$$

$$= \frac{1.2}{3} \quad \text{Simplify}$$

$$= 0.4$$

Linear equation: $y = 0.4x$

Substitute the given x-values to find y:

i. $y = 0.4(1) = 0.4$ A

ii. $y = 0.4(4) = 1.6$ A

c. Find the given values on the y-axis, and identify the corresponding x-value on the line. (Answers: i. 2 V ii. 5 V)

Alternatively, use the linear equation:

i. $0.8 = 0.4x \rightarrow x = \frac{0.8}{0.4} = 2$ V

ii. $2 = 0.4x \rightarrow x = \frac{2}{0.4} = 5$ V

Practice

1. Graph the equation: $y = 3x - 5$ using x-values between -2 and 2
2. Graph the equation: $5x + 4y = 20$
3. The speed of a car at various times is shown in the table.

Time (s)	0	3	6	9	12
Speed (km/h)	0	15	30	45	60

   a. Show this information on a graph using a scale of 1 cm to 1 second on the horizontal axis and a scale of 2 cm to 10 km/h on the vertical axis.
   b. Find the speed of the car after: i. 5 seconds ii. 8 seconds
   c. Find the time when the speed of the car is: i. 40 km/h ii. 65 km/h

4. Find the gradient of the line joining the points:
   a. $(2, -3)$ and $(4, 7)$
   b. $(11, 3)$ and $(6, 5)$
5. The gradient of the line joining the points $(5, 7)$ and $(9, m)$ is 2. Find the value of m.

Lesson Title: Applications of linear functions	**Theme:** Algebra
Practice Activity: PHM4-L022	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve problems involving linear functions.

Overview

This lesson uses information from the previous lesson to solve problems on linear functions. The following is additional information on lines that can be used to solve problems in this lesson.

Linear equations can be given by the formula $y - y_1 = m(x - x_1)$ where m is the gradient and (x_1, y_1) is a specific point on the line. (x, y) is a general point on the line.

Using this formula, we can find the equation of a line if we are given **either** of the following:

- The gradient m and a given point (x_1, y_1)
- Two given points (x_1, y_1) and (x_2, y_2)

To find the equation of a line given the gradient and a point, substitute the values of m, x_1, and y_1 into the formula $y - y_1 = m(x - x_1)$. Simplify and write the equation in slope-intercept form.

To find the equation of a line given two points, use the formula for gradient ($m = \frac{y_2 - y_1}{x_2 - x_1}$) to find the gradient. Then, follow the process above, using the gradient and **one** of the points on the line.

Solved Examples

1. Find the equation of the straight line passing through the points $(-1, -1)$ and $(3, 7)$.
   Solution:
   Let $(x_1, y_1) = (-1, -1)$ and $(x_2, y_2) = (3, 7)$

   Find the gradient:

m	$= \frac{7-(-1)}{3-(-1)}$	Substitute into the formula for m
	$= \frac{7+1}{3+1}$	Simplify
	$= \frac{8}{4} = 2$	

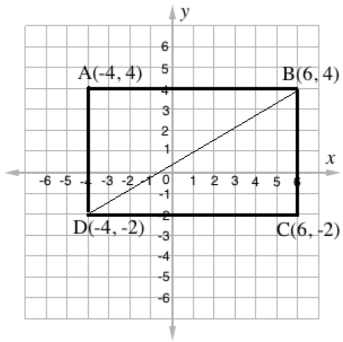
Find the equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Equation of a straight line} \\ y - 7 &= 2(x - 3) && \text{Substitute } m = 2 \text{ and one point, } (3, 7) \\ y - 7 &= 2x - 6 \\ y &= 2x - 6 + 7 && \text{Transpose } -7 \\ y &= 2x + 1 && \text{Equation of the line} \end{aligned}$$

2. Using a scale of 2 cm to 1 unit on both axes, draw a Cartesian plane with interval $-6 \leq x \leq 6$ and $-6 \leq y \leq 6$
   a. On the plane, draw rectangle ABCD with vertices $A(-4, 4)$, $B(6, 4)$, $C(6, -2)$, $D(-4, -2)$.
   b. Determine the equation of the line that passes through points B and D.

Solutions:

a. Draw the Cartesian plane and rectangle as shown on the right. Note that this is not to scale. Your tick marks should be 2 cm apart.



b. To determine the equation of the line, use the 2 points $B(6, 4)$ and $D(-4, -2)$. Find the gradient of the line and use the formula $y - y_1 = m(x - x_1)$ to write the linear equation.

Gradient:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} && \text{Substitute into the formula for } m \\ &= \frac{-2-4}{-4-6} && \text{Simplify} \\ &= \frac{-6}{-10} \\ m &= \frac{6}{10} = \frac{3}{5} \end{aligned}$$

Linear equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) && \text{Equation of a straight line} \\ y - 4 &= \frac{3}{5}(x - 6) && \text{Substitute } m = \frac{6}{10} \text{ and one point, } (6, 4) \\ y - 4 &= \frac{3}{5}x - \frac{18}{5} \\ y &= \frac{3}{5}x - \frac{18}{5} + 4 && \text{Transpose } -4 \\ y &= \frac{3}{5}x - \frac{18}{5} + \frac{20}{5} \\ y &= \frac{3}{5}x + \frac{2}{5} && \text{Equation of the line} \end{aligned}$$

3. Using a scale of 2 cm to 1 unit on both axes, and on the same Cartesian plane, draw the graphs of $y + \frac{3}{4}x = \frac{1}{2}$ and $2y - x = 6$. Identify the point of intersection of the 2 lines

Solution:

Draw the graphs of the lines using the method you prefer. You may create a table of values for each. However, it often saves time to write the equation in the form $y = mx + c$, and use the gradient (m) and y-intercept (c) to draw the graph.

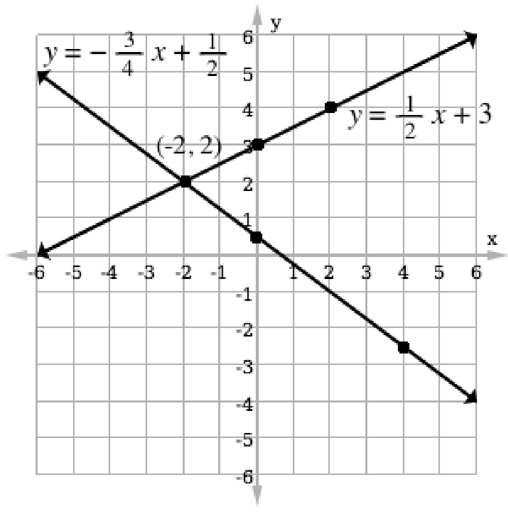
Write each equation in the form $y = mx + c$:

$$
\begin{aligned}
y + \tfrac{3}{4}x &= \tfrac{1}{2} \\
y &= -\tfrac{3}{4}x + \tfrac{1}{2}
\end{aligned}
\qquad\qquad
\begin{aligned}
2y - x &= 6 \\
2y &= x + 6 \\
y &= \tfrac{1}{2}x + 3
\end{aligned}
$$

For the first equation, we have $m = -\frac{3}{4}$ and $c = \frac{1}{2}$. Use this to graph the line. Identify $\frac{1}{2}$ on the positive y-axis. From there, count 3 in the negative y-direction (down) and 4 in the positive x-direction (right). Plot this point, which is $(4, -2\frac{1}{2})$. Connect $(0, \frac{1}{2})$ and $(4, -2\frac{1}{2})$ to form the line, $y = -\frac{3}{4}x + \frac{1}{2}$.

Follow the same process for the second equation. We have $m = \frac{1}{2}$ and $c = 3$. Identify 3 on the positive y-axis. From there, count 1 in the positive y-direction (up) and 3 in the positive x-direction (right). Plot this point, which is $(2, 4)$. Connect $(0, 3)$ and $(2, 4)$ to form the line, $y = \frac{1}{2}x + 3$.

Identify the point where the lines intersect, which is $(-2, 2)$.



Practice

1. Determine the equation of a straight line whose gradient is 3 and that passes through the point $(2, 8)$.
2. Find the equation of the straight line passing through the point $(1, 3)$ and $(-2, 5)$.
3. Using a scale of 2 cm to 1 unit on both axes, draw a Cartesian plane with interval $-6 \leq x \leq 6$ and $-6 \leq y \leq 6$
   a. On the plane, draw square ABCD with vertices $A(-5,5)$, $B(1, 5)$, $C(1, -1)$, $D(-5, -1)$.
   b. Determine the equations of the two lines that pass through the diagonals of the square.
4. Using a scale of 2 cm to 1 unit on both axes, and on the same Cartesian plane, draw the graphs of $y + \frac{1}{3}x - 3 = 0$ and $2y - x = 1$. Identify the point of intersection of the 2 lines.

Lesson Title: Distance and mid-point formulae	Theme: Algebraic Processes
Practice Activity: PHM4-L023	**Class:** SSS 4

Learning Outcomes

By the end of the lesson, you will be able to:

1. Identify and apply the distance formula to find the distance between one point and another on a line.
2. Identify and apply the mid-point formula to find the mid-point of a line.

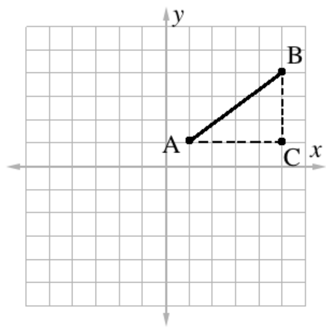
Overview

Distance formula:

The distance formula is related to Pythagoras' Theorem. For any points A(x_1, y_1) and B(x_2, y_2), we can use the distance formula below to calculate the distance between them using their coordinates.

$$|AB| = \sqrt{(x_2 - x_2)^2 + (y_2 - y_1)^2}$$

for any points A(x_1, y_1) and B(x_2, y_2)

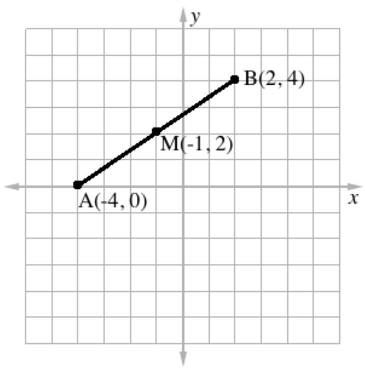


Mid-point Formula:

The mid-point is the point that is **exactly** mid-way, or in the middle, of two other points. The mid-point of two points (x_1, y_1) and (x_2, y_2) is the point M found by the following formula:

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

In the graph below, M is the mid-point of line AB.



Solved Examples

Distance:

1. Find the length of the line joining L$(-1, 8)$ and M$(3, 11)$

   Solution:

   Use the distance formula:

$$
\begin{aligned}
|LM| &= \sqrt{\left(3-(-1)\right)^2 + (11-8))^2} && \text{Substitute } x\text{- and } y\text{-values} \\
&= \sqrt{(3+1)^2 + (3)^2} && \text{Simplify} \\
&= \sqrt{4^2 + 3^2} \\
&= \sqrt{16+9} \\
&= \sqrt{25} \\
&= 5
\end{aligned}
$$

2. The coordinates of two points A and B are $\left(3, 1\frac{1}{2}\right)$ and $\left(2\frac{1}{2}, 5\right)$, respectively. Find the shortest distance between A and B.

   Solution:

   Note that the shortest distance between 2 points is the line connecting them.

   Use the distance formula:

$$
\begin{aligned}
|AB| &= \sqrt{\left(3-2\frac{1}{2}\right)^2 + (1\frac{1}{2}-5))^2} && \text{Substitute } x\text{- and } y\text{-values} \\
&= \sqrt{\left(\frac{1}{2}\right)^2 + (-3\frac{1}{2})^2} && \text{Simplify} \\
&= \sqrt{\frac{1}{4} + \frac{49}{4}} \\
&= \sqrt{\frac{50}{4}} \\
&= \frac{5\sqrt{2}}{2}
\end{aligned}
$$

3. Find the length of the line joining $(3a-5, 2b)$ and $(4a-5, 2b+7)$

   Solution:

$$
\begin{aligned}
|AB| &= \sqrt{\left((4a-5)-(3a-5)\right)^2 + ((2b+7)-(2b))^2} && \text{Substitute } x\text{- and } y\text{-values} \\
&= \sqrt{(4a-5-3a+5)^2 + (2b+7-2b)^2} \\
&= \sqrt{(a)^2 + (7)^2} && \text{Simplify} \\
&= \sqrt{a^2 + 49}
\end{aligned}
$$

4. Find the mid-point of line PQ if P is $(2, 5)$ and Q is $(6, 9)$.

   Solution:

   Use the mid-point formula:

$$
\begin{aligned}
M &= \left(\frac{6+2}{2}, \frac{9+5}{2}\right) && \text{Substitute } x\text{- and } y\text{-values} \\
&= \left(\frac{8}{2}, \frac{14}{2}\right) && \text{Simplify}
\end{aligned}
$$

$$= \quad (4, 7)$$

5. Find the mid-point M between $(3, 2)$ and $(-7, 16)$.
   Solution:

$$M = \left(\frac{-7+3}{2}, \frac{16+2}{2}\right) \qquad \text{Substitute } x\text{- and } y\text{-values}$$
$$= \left(\frac{-4}{2}, \frac{18}{2}\right) \qquad \text{Simplify}$$
$$= (-2, 9)$$

6. Find the value of l if $(5, -13)$ is the mid-point between P$(7, 8)$ and Q$(l, -34)$.

$$M = (5, -13) \qquad \leftarrow \text{Use this fact}$$

$$(5, -13) = \left(\frac{7+l}{2}, \frac{8+(-34)}{2}\right) \qquad \text{Apply the mid-point formula}$$
$$(5, -13) = \left(\frac{7+l}{2}, \frac{-26}{2}\right) \qquad \text{Simplify}$$
$$(5, -13) = \left(\frac{7+l}{2}, -13\right)$$

Note that the y-coordinates already match. We know the x-coordinates are equal to each other too. Set them equal and solve for l:

$$5 = \frac{7+l}{2}$$
$$2(5) = 2\left(\frac{7+l}{2}\right) \qquad \text{Multiply both sides by 2}$$
$$10 = 7 + l$$
$$10 - 7 = l \qquad \text{Transpose 7}$$
$$3 = l$$
$$l = 3$$

Practice

1. Find the length of line joining the following pairs of points:
   a. (5, 2), (2, 11)
   b. (3, 1), (2, 0)
   c. (-1, 4), (2, 6)
   d. (0, p), (p, 0)
   e. (p, $2p + 2$), ($p - 2$, $3p + 2$)
2. Find the length of the line from the origin to (3, 4).
3. By using Pythagoras' theorem, show that the triangle PQR is right-angled where P is (-4, -2), Q is (4, 2) and R is (2, 6).
4. Find the coordinates of the mid-point of the line joining:
   a. (3, 7), (5, 9)
   b. (4, -3), (6, 1)
   c. ($3p$, q), (p, $5q$).
5. Find the value of c. if (-2, 4) is the mid-point of the line joining A(3, 2) and B(-7, c).

Lesson Title: Graphing and interpreting quadratic functions	**Theme:** Algebraic Processes
Practice Activity: PHM4-L024	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to graph quadratic functions, and identify the solutions, and maximum or minimum.

Overview

A “quadratic equation” is given in a way that allows you to solve for a variable and does not have a y-value (Example: $x^2 + 4x + 3 = 0$). A “quadratic function” has a y-value and can be graphed (Example: $y = x^2 + 4x + 3$). Quadratic equations and functions have a term with a variable raised to the power 2 (x^2).

Quadratic equations can be solved by graphing the related quadratic function. The graph of a quadratic function is a parabola. The **solutions** (or **roots**) of a quadratic equation are the values of x where the parabola crosses the x-axis.

The **maximum** or **minimum** of a quadratic function is the turning point of the graphed parabola. The minimum is the lowest point of a parabola that opens up. The maximum is the highest point of a parabola that opens down. The minimum or maximum can be given as an ordered pair, (x, y).

The **line of symmetry** is a vertical line that passes through the maximum or minimum. It is parallel to the y-axis. It can be written as a linear equation with one variable, x (Example: $x = 2$).
Examples:

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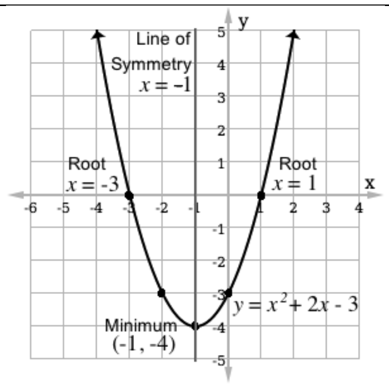

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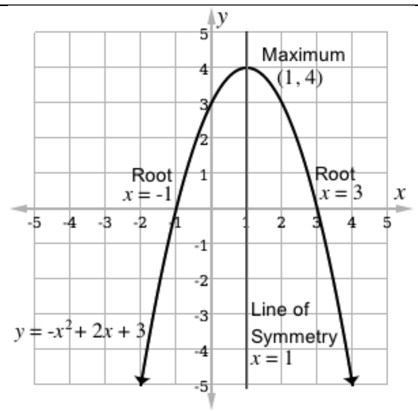

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- The graph of $y = x^2 + 2x - 3$ opens up. It has a line of symmetry at $x = -1$, and a minimum at $(-1, -4)$.
- The solutions of $x^2 + 2x - 3 = 0$ are the roots, $x = -3$ and $x = 1$.

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- The graph of $y = -x^2 + 2x + 3$ opens down. It has a line of symmetry at $x = 1$, and a maximum at $(1, 4)$.
- The solutions of $-x^2 + 2x + 3 = 0$ are the roots, $x = -1$ and $x = 3$.

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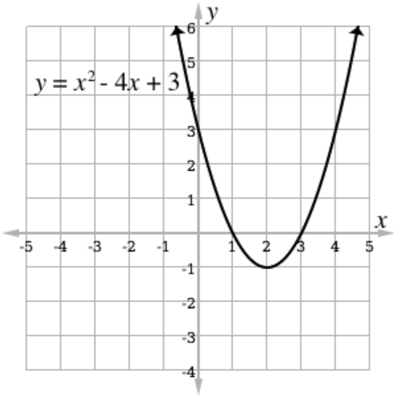
Solved Examples

1. The graph below shows the relation $y = x^2 - x - 6$ for the interval $-2 \leq x \leq 3$. Use it to answer the following questions:
   a. What is the minimum value of $y = x^2 - x - 6$?
   b. What is the solution set of the equation $x^2 - x - 6 = 0$?
   c. What is the equation of the line of symmetry?



Solutions:
   a. The minimum is $(2, -1)$.
   b. The solution set consists of the x-values where the parabola crosses the x-axis, $x = 1, 3$.
   c. The line of symmetry is the vertical line that passes through the minimum, $x = 2$.

2. Graph the quadratic function $y = -x^2 + x + 2$ for the interval $-2 \leq x \leq 3$. Use it to answer the following questions:
   a. What is the maximum value of $y = -x^2 + x + 2$?
   b. What is the solution set of the equation $-x^2 + x + 2 = 0$?
   c. What is the equation of the line of symmetry?

   Solutions:

   Step 1. Graph the function using a table of values with x-values from -2 to 3:

x	-2	-1	0	1	2	3
y	-4	0	2	2	0	-4

$$\begin{aligned} y &= -(-2)^2 + (-2) + 2 \\ &= -4 - 2 + 2 \\ &= -4 \end{aligned}$$

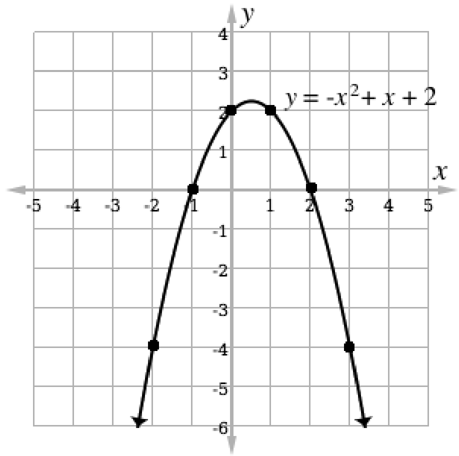
$$\begin{aligned} y &= -(-1)^2 + (-1) + 2 \\ &= -1 - 1 + 2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} y &= -(0)^2 + (0) + 2 \\ &= 0 + 0 + 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} y &= -(2)^2 + (2) + 2 \\ &= -4 + 2 + 2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} y &= -(3)^2 + (3) + 2 \\ &= -9 + 3 + 2 \\ &= -4 \end{aligned}$$

$$
\begin{aligned}
y &= -(1)^2 + (1) + 2 \\
&= -1 + 1 + 2 \\
&= 2
\end{aligned}
$$



Step 2. Use the graph to answer the questions:

a. The maximum is at $x = 0.5$. To write this as an ordered pair, substitute $x = 0.5$ into the quadratic equation and solve for y:

$$
\begin{aligned}
y &= -x^2 + x + 2 \\
&= -0.5^2 + 0.5 + 2 \\
&= -0.25 + 0.5 + 2 \\
&= 2.25
\end{aligned}
$$

The maximum point is $(0.5, 2.25)$.

b. The solution set consists of the x-values where the parabola crosses the x-axis, $x = -1, 2$.

c. The line of symmetry is the vertical line that passes through the maximum, $x = 0.5$.

Practice

1. Graph the function $y = x^2 + 5x - 6$ using the interval $-5 \leq x \leq 1$. From your graph:
   a. Find the solution set of the equation $x^2 + 5x - 6 = 0$
   b. Write down the minimum of the function.
   c. What is the equation of the line of symmetry?

2. Draw the graph of the function $y - x^2 + 9x - 20 = 0$ using the interval $3 \leq x \leq 7$. From your graph:
   a. Find the roots of the equation $x^2 - 9x + 20 = 0$.
   b. Write down the minimum value of the function $y = x^2 - 9x + 20$.
   c. Find the approximate value of y when $x = 1.5$.
   d. What is the equation of the line of symmetry?

3. Using a suitable scale, draw the graph of the quadratic function $y = -x^2 - 3x + 10$ for the interval $-3 \leq x \leq 2$. From your graph,
   a. Find the roots of the equation $-x^2 - 3x + 10 = 0$
   b. Write down the maximum value of the function.

c. What is the equation of the line of symmetry?
d. Find the approximate value of y when $x = 2.1$

4. Copy and complete the following table of values for the relation $y = 10 + 6x - 3x^2$ for $-3 \leq x \leq 5$

x	-3	-2	-1	0	1	2	3	4	5
y		-14		10			1		

a. Use a scale of 2 cm to 1 unit on the x-axis and 2 cm to 5 units on the y-axis, draw the graph of the relation
b. From the graph, solve the equation $10 + 6x - 3x^2 = 0$.
c. What is the maximum of the function?
d. Find the equation of the axis of symmetry
e. Find the approximate value y when $x = 1.5$.

Lesson Title: Solving quadratic equations algebraically – Part 1	**Theme:** Algebraic Processes
Practice Activity: PHM4-L025	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to factorise and solve quadratic equations.

Overview

A quadratic equation is in the form $ax^2 + bx + c = 0$

We can find the solutions to quadratic equations in several ways:

- By graphing the related function
- Using factorisation
- Completing the square
- Using the quadratic formula

This lesson focuses on factorisation. To factorise quadratic equation $ax^2 + bx + c = 0$, find two numbers that multiply to give ac and add to give b. Rewrite the quadratic equation as two binomials. For example, consider the quadratic equation $x^2 + 2x - 3 = 0$. Factorisation involves rewriting the quadratic equation in the form $x^2 + 2x - 3 = (x + p)(x + q)$, where p and q are numbers that multiply to give $ac = -3$ and add to give $b = 2$.

To find the solutions (or "roots") of a quadratic equation, set each binomial equal to 0, and solve for x. Each value of x is a solution. It is possible for quadratic equations to have 1, 2, or no solutions.

Solved Examples

1. Solve the quadratic equation $x^2 + 3x - 10 = 0$ using factorisation.
   Solution:

$x^2 + 3x - 10 \;=\; (x + p)(x + q)$		Set up the factorisation
$p + q = 3$	$p \times q = -10$	Note that the values of p an q must be -2 and 5
$x^2 + 3x - 10 \;=\; (x - 2)(x + 5) = 0$		Substitute values of p and q
$x - 2 = 0$ $x = 2$	$x + 5 = 0$ $x = -5$	Set each binomial equal to 0 and solve for x. These are the roots.

2. Solve the quadratic equation $2x^2 + 5x - 12 = 0$ using factorisation.
Solution:

$2x^2 + 5x - 12$	$=$	$(2x + p)(x + q)$	Set up the factorisation
$p + q = 5$		$p \times q = 2 \times -12$ $= -24$	Note that the values of p an q must be -3 and 8
$2x^2 + 5x - 12$	$=$	$(2x - 3)(x + 4) = 0$	Substitute values of p and q
$2x - 3 = 0$ $x = \frac{3}{2}$		$x + 4 = 0$ $x = -4$	Set each binomial equal to 0 and solve for x.

3. Find the values of x for which $x^2 - 3x = -2$.
Solution:
Rewrite the equation so it is equal to zero: $x^2 - 3x + 2 = 0$

Factorise:

$x^2 - 3x + 2$	$=$	$(x + p)(x + q)$	Set up the factorisation
$p + q = -3$		$p \times q = 2$	Note that the values of p an q must be -2 and -1
$x^2 - 3x + 2$	$=$	$(x - 2)(x - 1) = 0$	Substitute values of p and q
$x - 2 = 0$ $x = 2$		$x - 1 = 0$ $x = 1$	Set each binomial equal to 0 and solve for x.

4. Use the method of factorisation to solve the equation $y - \frac{12}{y} = 1$
Solution:
This can be solved by factorisation, but we must first multiply throughout by y to eliminate the fraction.

$y(y) - y(\frac{12}{y})$	$=$	$y(1)$	Multiply throughout by y
$y^2 - 12$	$=$	y	
$y^2 - y - 12$	$=$	0	Write in standard form.
$y^2 - y - 12$	$=$	$(y + p)(y + q)$	Set up the factorisation
$p + q = -1$		$p \times q = -12$	Note that the values of p and q must be -4 and 3
$y^2 - y - 12$	$=$	$(y - 4)(y + 3) = 0$	Factorise the expression

$y - 4 = 0$ $y = 4$	$y + 3 = 0$ $y = -3$	Set each binomial equal to 0 and solve for y. These are the roots.

5. Find the smaller value of x which satisfies the equation: $x^2 - 4x + 3 = 0$.
   Solution:
   Complete the factorisation, then choose the smaller value.

$x^2 - 4x + 3 \quad = \quad (x + p)(x + q)$		Set up the factorisation
$p + q = -4$	$p \times q = 3$	Note that the values of p an q must be -3 and -1
$x^2 - 4x + 3 \quad = \quad (x - 3)(x - 1) = 0$		Substitute values of p and q
$x - 3 = 0$ $x = 3$	$x - 1 = 0$ $x = 1$	Set each binomial equal to 0 and solve for x. These are the roots.

   The smaller value of x is $x = 1$.

6. Find the equation whose roots are $x = 4$ and $x = -1$.
   Solution:
   This question requires us to work backwards. If the roots of an equation are $x = 4$ and $x = -1$, we know the 2 binomial factors are $(x - 4)$ and $(x + 1)$. Multiply these to find the equation:

$$(x - 4)(x + 1) = x^2 - 3x - 4 = 0$$

7. Solve using factorisation: $(x - 3)(x - 2) = 2(x - 2)$
   Solution:
   First rewrite the equation as a quadratic equation in standard form. Then, factorise and find the solutions.

$(x - 3)(x - 2)$	$=$	$2(x - 2)$	
$x^2 - 5x + 6$	$=$	$2x - 4$	Simplify both sides
$x^2 - 5x - 2x + 6 + 4$	$=$	0	Transpose $2x - 4$
$x^2 - 7x + 10$	$=$	0	Combine like terms
$x^2 - 7x + 10$	$=$	$(x - 5)(x - 2)$	Factorise

   Find the roots:

$x - 5 = 0$ $x = 5$	$x - 2 = 0$ $x = 2$

8. Solve for x: $\frac{1}{2x+1} + \frac{2}{x+1} = \frac{4}{3}$

Solution:

This can be solved by factorisation, but we must first multiply throughout by $3(2x+1)(x+1)$ to eliminate the fractions.

$3(x+1)+6(2x+1)$	$=$	$4(2x+1)(x+1)$	Multiply throughout by $3(x+1)(2x+1)$
$3x+3+12x+6$	$=$	$4(2x^2+3x+1)$	Simplify
$3x+3+12x+6$	=	$8x^2+12x+4$	
0	=	$8x^2-3x-5$	Write in standard form.
0	=	$(x-1)(8x+5)$	Factorise the expression

Find the roots:

$x-1=0$ $\quad$ $8x+5=0$

$x=1$ $\quad$ $x=-\frac{5}{8}$

Practice

Solve the following quadratic equations using the method of factorisation:

1. $x^2+x-12=0$
2. $x^2-7x-18=0$
3. $2x^2+11x+12=0$
4. $y^2-2by+b^2=0$
5. $m(1-m)=m(2m-1)$
6. $(x-2)(x+3)=(x-2)(4-x)$
7. $\frac{1}{y+1}+\frac{2}{y+2}=1$

Lesson Title: Solving quadratic equations algebraically – Part 2	Theme: Algebra
Practice Activity: PHM4-L026	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Solve a quadratic equation by completing the square.
2. Solve a quadratic equation using the quadratic formula.

Overview

Consider the equation $x^2 + 4x + 2 = 0$. We cannot use simple factorisation to solve this quadrilateral. We can solve it by "completing the square" or using the quadratic formula.

Completing the square

Expressions such as $(x + 2)^2$ are called perfect squares because they are squares of binominals. They are easy to expand and easy to factorise back into factors.

Completing the square changes a quadratic expression to the sum of a perfect square and a number.

Compare $x^2 + 4x + 2$ with the perfect square $(x + 2)^2 = x^2 + 4x + 4$. There is a difference of 2. Thus, we can rewrite $x^2 + 4x + 2$ using the perfect square:

$$\begin{aligned} x^2 + 4x + 2 &= x^2 + 4x + 4 - 2 \\ &= (x + 2)^2 - 2 \end{aligned}$$

After completing the square, the expression can be solved for x. This gives the solutions, or roots, of the quadratic equation.

Generally, every imperfect square in the form $ax^2 + bx + c$ can be written as $ax^2 + bx + c = (x + m)^2 + n$ where m and n are constants. $ax^2 + bx + c = x^2 + 2mx + m^2 + n$ gives formulae: $m = \frac{b}{2}$ and $m^2 + n = c$, when $a = 1$. If $a \neq 1$, then $m = \frac{b}{2a}$ and $n = \frac{c}{a} - \frac{b^2}{4a^2}$. These can be used to complete the square by finding m and n.

Quadratic formula

The quadratic formula is useful when the quadratic expression cannot be easily factorised. It uses the values of a, b and c from the standard form of the quadratic formula, $ax^2 + bx + c = 0$.

The quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

In solving problems using the formula, write down the value of a, b and c and substitute these values in the formula to find the roots of the equation.

Solved Examples

1. Find the roots of $x^2 + 4x + 3 = 0$ by completing the square.

   Solution:

   Note that in this equation, $a = 1, b = 4$ and $c = 3$.

   $x^2 + 4x + 3$ is an imperfect square, so write in the form:

   $$x^2 + 4x + 3 = (x + m)^2 + n$$
   $$x^2 + 4x + 3 = x^2 + 2mx + m^2 + n$$

   Set the second term, $2mx$, equal to b and solve for m. Alternatively, use the formula $m = \frac{b}{2}$.

   $$\begin{aligned} 4 &= 2m \\ \frac{4}{2} &= \frac{2m}{2} \\ 2 &= m \end{aligned}$$

   Set the constant terms equal and solve for n:

   $$\begin{aligned} 3 &= m^2 + n \\ 3 &= 2^2 + n \\ 3 &= 4 + n \\ 3 - 4 &= n \\ -1 &= n \end{aligned}$$

   Substitute $m = 2$ and $n = -1$ into the formula, $(x + m)^2 + n$.

   $$\begin{aligned} x^2 + 4x + 3 &= (x + m)^2 + n = 0 \\ &= (x + 2)^2 - 1 = 0 \end{aligned}$$

   Solve for x:

   $$\begin{aligned} (x + 2)^2 - 1 &= 0 \\ (x + 2)^2 &= 1 \\ \sqrt{(x + 2)^2} &= \sqrt{1} \\ x + 2 &= \pm 1 \\ x &= -2 \pm 1 \\ \text{Either } x &= -2 + 1 = -1 \\ \text{or } x &= -2 - 1 = -3 \end{aligned}$$

   Therefore, the roots of $x^2 + 4x + 3 = 0$ are $x = -1, -3$.

2. Find the roots of the quadratic equation by completing the square: $x^2 + 5x - 2 = 0$

   Solution:

   To complete the square, write $x^2 + 5x - 2$ in the form $(x + m)^2 + n$. Now you have $x^2 + 5x - 2 = (x + m)^2 + n = x^2 + 2mx + m^2 + n$

   Set the second term, $2mx$, equal to b and solve for m.

   $$\begin{aligned} 5 &= 2m \\ \frac{5}{2} &= m \end{aligned}$$

   Set the constant terms equal and solve for n:

   $$-2 = m^2 + n$$

$$-2 = \left(\frac{5}{2}\right)^2 + n$$
$$-2 = \frac{25}{4} + n$$
$$-2 - \frac{25}{4} = n$$
$$-\frac{33}{4} = n$$

Substitute $m = \frac{5}{2}$ and $n = -\frac{33}{4}$ into the formula, $(x+m)^2 + n$.

$$x^2 + 4x + 3 = (x+m)^2 + n = 0$$
$$= \left(x + \frac{5}{2}\right)^2 - \frac{33}{4} = 0$$

Solve for x:

$$\left(x + \frac{5}{2}\right)^2 - \frac{33}{4} = 0$$
$$\left(x + \frac{5}{2}\right)^2 = \frac{33}{4}$$
$$x + \frac{5}{2} = \pm\sqrt{\frac{33}{4}}$$
$$x = -\frac{5}{2} \pm \sqrt{\frac{33}{4}}$$

Either $x = -\frac{5}{2} + \frac{\sqrt{33}}{2} = \frac{-5+\sqrt{33}}{2}$

or $x = -\frac{5}{2} - \frac{\sqrt{33}}{2} = \frac{-5-\sqrt{33}}{2}$

3. Use the quadratic formula to solve $x^2 + 13x + 22 = 0$.

Solution:

Note that $a = 1, b = 13, c = 22$. Substitute these values and evaluate:

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Formula

$= \frac{-13 \pm \sqrt{13^2 - (4)(1)(22)}}{2(1)}$ Substitute $a = 1, b = 13, c = 22$

$= \frac{-13 \pm \sqrt{169 - 88}}{2}$ Evaluate

$= \frac{-13 \pm \sqrt{81}}{2}$

$= \frac{-13 \pm 9}{2}$

$x = \frac{-13+9}{2}$ or $\frac{-13-9}{2}$

$= -\frac{4}{2}$ or $\frac{-22}{2}$

$= -2$ or -11

4. Use the quadratic equation formula to solve $5m^2 - 2m - 3 = 0$.

Solution:

Note that $a = 5, b = -2, c = -3$.

$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-(-2)\pm\sqrt{(-2)^2-(4)(5)(-3)}}{2(5)}$$
$$= \frac{2\pm\sqrt{4+60}}{10}$$
$$= \frac{2\pm\sqrt{64}}{10}$$
$$m = \frac{2+8}{10} \text{ or } \frac{2-8}{10}$$
$$= \frac{10}{10} \text{ or } -\frac{6}{10}$$
$$= 1 \text{ or } -\frac{3}{5}$$

5. Solve $7y^2 - 3y - 10 = 0$ using the quadratic formula.
   Solution:
   Note that $a = 7, b = -3, c = -10$.
$$y = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$= \frac{-(-3)\pm\sqrt{(-3)^2-(4)(7)(-10)}}{2(7)}$$
$$= \frac{3\pm\sqrt{9+280}}{14}$$
$$= \frac{3\pm\sqrt{289}}{14}$$
$$y = \frac{3+17}{14} \text{ or } \frac{3-17}{14}$$
$$= \frac{20}{14} \text{ or } -\frac{14}{14}$$
$$= \frac{10}{7} \text{ or } -1$$

Practice

1. Find the roots of $3x^2 - 8x + 2 = 0$ using the completing the square method.
2. Find the roots of the quadratic equation by completing the square: $2x^2 - 10x + 7 = 0$.
3. Find the roots of the quadratic equations by completing the square:
   a. $6x^2 + 13x + 6 = 0$
   b. $2a^2 + 7a - 3 = 0$
4. Use the quadratic formula to solve the following equations:
   a. $y^2 + 12y + 11 = 0$
   b. $x^2 - 16x + 28 = 0$
   c. $m^2 - 5m = 6$
   d. $y^2 - 9y + 14 = 0$
   e. $4m^2 - 17m + 18 = 0$

Lesson Title: Problem solving with quadratic equations	**Theme:** Algebra
Practice Activity: PHM4-L027	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve problems that lead to quadratic equations.

Overview

This lesson is on solving various quadratic equations problems, including word problems. Follow the regular process for solving word problems:

1. Read through the problem carefully and know what is being asked.
2. Choose variables to represent unknown numbers.
3. Write an algebraic expression for the problem.
4. Solve the problem.
5. Check your work.

If a problem does not specify what method to use, you may use any method to solve the quadratic equation.

Solved Examples

1. Find two numbers whose difference is 3 and whose product is 180.
   Solution:
   Let the smaller of the two numbers be x. Then the larger number is $x + 3$, and their product is $x(x + 3)$.

   Write an equation and evaluate. Factorisation is used below, but you may use any method you prefer.

$x(x+3)$	$=$	180	Equation
$x^2 + 3x$	$=$	180	Expand brackets
$x^2 + 3x - 180$	$=$	0	
$(x+15)(x-12)$	$=$	0	Factorise
x	$=$	$-15, 12$	

   The smaller number (x) has 2 possible values: -15 and 12. Find the value of the bigger number for each of them:

   $x = -15$ → $-15 + 3 = -12$
   $x = 12$ → $12 + 3 = 15$

   Therefore, the 2 numbers are -15 and -12, or 12 and 15.

   Check your work:

   $-15 \times -12 = 180$ and $15 \times 12 = 180$

2. The length of a rectangular table top is 20 cm more than the width. The area of the table top is 3,500 cm^2. Find the length and width of the table top.
Solution:
If the width is w, then the length is $l = w + 20$. Recall that the formula for the area of a rectangle is $A = lw$. Use this to solve for w:

$$\begin{aligned} A = lw &= 3{,}500 \\ w(w+20) &= 3{,}500 && \text{Equation} \\ w^2 + 20w &= 3{,}500 && \text{Expand brackets} \\ w^2 + 20w - 3500 &= 0 \\ (w+70)(w-50) &= 0 && \text{Factorise} \\ w &= -70, 50 \text{ cm} \end{aligned}$$

Note that the length cannot be negative. Therefore, the width must be 50 cm. Use this to find the length:

$$l = w + 20 = 50 + 20 = 70 \text{ cm}$$

Check your work:

$$A = lw = 70 \times 50 = 3{,}500 \text{ cm}^2$$

3. The area of a rectangle is 20 m^2, and its perimeter is 24 m. Find the length and width of the rectangle.
Solution:
Recall that the formula for area of a rectangle is $A = lw$, and the formula for perimeter of a rectangle is $P = 2l + 2w$. Therefore, we have the simultaneous equations:

$$\begin{aligned} 20 &= lw && (1) \\ 24 &= 2l + 2w && (2) \end{aligned}$$

Solve by substitution. Solve Equation (2) for either l or w, and substitute the result into Equation (1).

$$\begin{aligned} 24 &= 2l + 2w && (2) \\ 24 - 2w &= 2l \\ \tfrac{24-2w}{2} &= \tfrac{2l}{2} \\ 12 - w &= l \end{aligned}$$

Substitute $l = 12 - w$ into Equation (1) and solve for w:

$$\begin{aligned} 20 &= lw && (1) \\ 20 &= (12 - w)w \\ 20 &= 12w - w^2 \\ w^2 - 12w + 20 &= 0 && \text{Quadratic equation} \\ (w-10)(w-2) &= 0 \\ w &= 10, 2 \end{aligned}$$

Substitute these values into Equation (2) to find corresponding values for l:

$$l = 12 - w = 12 - 10 = 2$$
$$l = 12 - w = 12 - 2 = 10$$

The solution is $w = 10$ and $l = 2$, or $w = 2$ and $l = 10$. We know that length is longer than width, so the answer is $w = 2$ and $l = 10$.

4. The product of 2 consecutive positive even numbers is 168. By constructing a quadratic equation and solving it, find the two numbers.
   Solution:
   Let the numbers be x and y. If x is the smaller number, than $y = x + 2$.
   Multiply and solve for x:

$$
\begin{aligned}
x(x+2) &= 168 && \text{Equation} \\
x^2 + 2x &= 168 && \text{Expand brackets} \\
x^2 + 2x - 168 &= 0 && \\
(x+14)(x-12) &= 0 && \text{Factorise} \\
x &= -14, 12 &&
\end{aligned}
$$

   We are given in the problem that both numbers are even. Therefore, consider only $x = 12$. Find the corresponding value of y:

$$y = x + 2 = 12 + 2 = 14$$

   Thus, the 2 numbers are 12 and 14.

   Check your work: $12 \times 14 = 168$

Practice

1. The product of 2 consecutive negative odd numbers is 143. Find the two numbers.
2. The area of a rectangle is 30 m^2, and its perimeter is 22 m. Find the length and width of the rectangle.
3. Two rectangles have the same area of 36 m^2. The second rectangle is 3 m shorter and 1 metre wider than the first rectangle. Find the length and width of the first rectangle.
4. Find two numbers whose difference is 7 and whose product is 144.
5. The length of a rectangular garden is 10 m longer than the width. If the area of the garden is 96 cm^2, find its length and width.

Lesson Title: Simultaneous linear equations	Theme: Algebraic Processes
Practice Activity: PHM4-L028	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve simultaneous linear equations using elimination, substitution, and graphing.

Overview

Simultaneous linear equations are 2 equations that are solved at the same time (or simultaneously), and which have the same answer. We must solve for 2 unknown variables (usually x and y), and the two results should satisfy both equations.

For example, this is a set of simultaneous equations:

$$2x + 5y = -1 \qquad (1)$$
$$8x + 5y = 11 \qquad (2)$$

This is the answer to the simultaneous equations: $x = 2, y = -1$, or $(2, -1)$
There are 3 ways to solve simultaneous equations: elimination, substitution, and graphing.

To solve by **elimination:**

- Subtract one equation from the other so that one of the variables (x or y) is eliminated.
- If one of the variables has the same coefficient in the two equations, it will easily be canceled by subtraction.
- If neither of the variables has the same coefficient, you will need to multiply the equations throughout so that the coefficients are the same before subtracting.
- Check your answers by substituting the values for x and y into the original equations.

To solve by **substitution:**

- To solve using the method of substitution, we must change the subject.
- We should choose one of the given equations and make one of the variables the subject of the other one.
- If one of the variables does not have a coefficient, it is easiest to solve for that variable.
- After changing the subject, we substitute the expression into the other linear equation.

To solve by **graphing:**

- We can solve a set of simultaneous equations by graphing both lines. The solution is the point where the lines intersect. At this point, the x-value and y-value satisfy both of the equations.

Solved Examples

1. Solve the following simultaneous equations using the elimination method:

$$\begin{aligned} 6x+y &= 9 \\ -4x+y &= -11 \end{aligned}$$

Solution (using elimination):

$6x+y$	$=$	9	(1)	
$-(-4x+y$	$=$	$-11)$	(2)	Subtract equation (2) from equation (1).
$10x+0$	$=$	20		**Note:** The negative sign will change all the signs across the bracket
$10x$	$=$	20		
$\frac{10x}{10}$	$=$	$\frac{20}{10}$		Divide both sides of the equation by the co-efficient of y (which is 2)
x	$=$	2		

Substitute the value of $x = 2$ in either equation 1 or 2 to find the value of y:

$6x+y$	$=$	9	(1)	
$6(2)+y$	$=$	9		Substitute $x = 2$ in equation (1)
$12+y$	$=$	9		
Y	$=$	$9-12$		Transpose 12
y	$=$	-3		

The solution is therefore $x = 2, y = -3$, or $(2, -3)$.

Check the answer by substituting $x = 2, y = -3$ into equation 2. The left-hand side (LHS) should equal the right-hand side (RHS):

$-4x+y$	$=$	-11	(2)	
$-4(2)-3$	$=$	-11		Substitute $x = 2, y = -3$ in equation (2)
$-8-3$	$=$	-11		
-11	$=$	-11		
LHS	$=$	RHS		

2. Solve the following simultaneous equations by the substitution method:

$$\begin{aligned} 6x + y &= 9 \quad (1) \\ y - 4x &= -11 \quad (2) \end{aligned}$$

Solution (using substitution):

$$y - 4x = -11 \quad (2)$$

$$y = 4\text{x} - 11 \quad \text{Change the subject of equation (2)}$$

$$6x + (4x - 11) = 9 \quad (1) \quad \text{Substitute equation (2) into equation (1)}$$

$$6x + 4x - 11 = 9 \quad \text{Simplify the left-hand side}$$

$$10x - 11 = 9$$

$$10x = 9 + 11 \quad \text{Transpose } -11$$

$$10x = 20$$

$$\frac{10x}{10} = \frac{20}{10} \quad \text{Divide throughout by } 10$$

$$x = 2$$

Substitute the value $x = 2$ into the formula for y, $y = 4x - 11$.

$$y = 4(2) - 11 \quad \text{Substitute } x = 2 \text{ in the formula for } y$$

$$y = 8 - 11$$

$$y = -3$$

Thus, the solution is $x = 2$, $y = -3$ or $(2, -3)$.

3. Solve the following simultaneous equations by the graphical method:

$$\begin{aligned} 2x - y &= 1 \quad (1) \\ 3x + y &= 4 \quad (2) \end{aligned}$$

Solution (using graphing):

Generate Cartesian coordinates for equation (1). Note: since the equation is linear, three points are sufficient to draw the graph.

$$2x - y = 1 \quad (1)$$

$$y = 2x - 1 \quad \text{Make } y \text{ the subject}$$

$$y = 2(-1) - 1 \quad \text{Substitute each value of } x \text{ and solve for } y$$

$$y = -3$$

$$y = 2(0) - 1$$

$$y = -1$$

$$y = 2(1) - 1$$

$$y = 1$$

Equation (1)			
x	-1	0	1
y	-3	-1	1

Generate a table of values for equation (2)

$$\begin{aligned} 3x + y &= 4 && (2) \\ y &= -3x + 4 && \text{Make y the subject} \end{aligned}$$

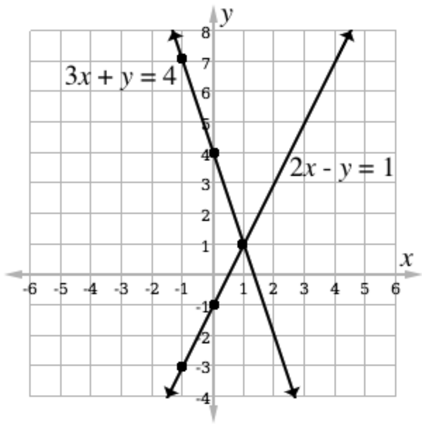
$$\begin{aligned} y &= -3(-1) + 4 && \text{Substituting each value of } x \text{ and solve for } y \\ y &= 7 \end{aligned}$$

$$\begin{aligned} y &= -3(0) + 4 \\ y &= 4 \end{aligned}$$

$$\begin{aligned} y &= -3(1) + 4 \\ y &= 1 \end{aligned}$$

Equation (2)			
x	-1	0	1
y	7	4	1

Graph both lines on a Cartesian plane:



The solution is the point of intersection, $(1, 1)$.

Practice

1. Solve the following simultaneous equations using the elimination method.
   a. $6x + y = 9$ and $4x - y = 11$
   b. $\frac{2x-3y}{4} = \frac{x-y}{3} = 1$
2. Solve the following simultaneous equations using the substitution method.
   a. $2x + 3y = 5$ and $x + 2y = 4$
   b. $4x + 7y = 16$ and $6x + 5y = 13$
3. Solve the following simultaneous equations using the graphical method.
   a. $2x - y = 2$ and $3x + 2y = 17$
   b. $y = 4x + 3$ and $y + x - 3 = 0$

Lesson Title: Applications of simultaneous linear equations	Theme: Algebraic Processes
Practice Activity: PHM4-L029	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve word problems leading to simultaneous linear equations.

Overview

Algebra is often used to solve word problems. We often need to identify the unknown variables in word problems, and use letters to represent them. In some cases, a word problem will give two equations with the same variables. These are simultaneous equations, and they can be solved using either elimination or substitution.

Follow this process to solve simultaneous equation word problems:

1) Identify the unknown variables
2) Create simultaneous linear equations
3) Solve using the method you prefer

There are various problems on the WASSCE exam that do not tell a story, but require you to have an understanding of simultaneous equations. Solved Example 3 is one example of such a problem.

Solved Examples

1. The sum of the digits of a number is 8. When the digits are interchanged, the number is increased by 18. Find the number.
   Solution:

   Step 1. Identify the unknown variables:

   Let x be the tens digit and y the unit digit of the number. The number is therefore xy.

   Step 2. Create the linear equations:

   Sum of the digits is 8

   $$x + y = 8 \qquad (1)$$

   The number is xy. When the digits are interchanged, the number becomes yx and increases by 18. This gives $yx = xy + 18$. Rewrite the equation:

   $$yx = xy + 18$$

$10y + x$	=	$10x + y + 18$		Expand the numbers
$-9x + 9y$	=	18	(2)	Transpose $10x$ and y

Step 3. Solve by either elimination or substitution:

Elimination:

$$
\begin{aligned}
x + y &= 8 && (1)\\
-9x + 9y &= 18 && (2)\\
\downarrow \qquad & \qquad \downarrow\\
9x + 9y &= 72 && (1)\times 9\\
-9x + 9y &= 18 && (2)\\
0 + 18y &= 90\\
y &= 5\\
\\
x + 5 &= 8 && (1)\\
x &= 8 - 5\\
x &= 3
\end{aligned}
$$

Substitution:

$$
\begin{aligned}
y &= 8 - x && (1)\\
\\
-9x + 9(8 - x) &= 18 && (2)\\
-9x + 72 - 9x &= 18\\
-18x &= -54\\
x &= 3\\
\\
y &= 8 - 3 && (1)\\
y &= 5
\end{aligned}
$$

Answer: $x = 3, \;\; y = 5$

The number (xy) is therefore 35.

2. Five cups and 3 plates cost Le 19,000.00. Four cups and 6 plates cost Le 26,000.00. What is the cost of a cup and a plate?

Solution:

Step 1. Identify the unknown variables:

c: cost of a cup
p: cost of a plate

Step 2. Create the linear equations:

5 cups and 3 plates cost Le 19,000.00:	$5c + 3p = 19{,}000$	(1)
4 cups and 6 plates cost Le 26,000.00:	$4c + 6p = 26{,}000$	(2)

Step 3. Solve by either elimination or substitution:

Elimination:

$$\begin{aligned} 5c + 3p &= 19{,}000 && (1) \\ 4c + 6p &= 26{,}000 && (2) \\ \downarrow & \quad\quad \downarrow \\ 10c + 6p &= 38{,}000 && (1)\times 2 \\ -(4c + 6p &= 26{,}000) && (2) \\ 6c + 0 &= 12{,}000 \\ c &= 2{,}000 \\ 5c + 3p &= 19{,}000 && (1) \\ 5(2{,}000) + 3p &= 19{,}000 \\ p &= 3{,}000 \end{aligned}$$

Substitution:

$$\begin{aligned} c &= 3{,}800 - \tfrac{3}{5}p && (1) \\ 4\left(3{,}800 - \tfrac{3}{5}p\right) + 6p &= 26{,}000 && (2) \\ 4(19{,}000 - 3p) + 30p &= 130{,}000 \\ 38{,}000 - 6p + 15p &= 65{,}000 \\ 9p &= 27{,}000 \\ p &= 3{,}000 \\ c &= \tfrac{19{,}000}{5} - \tfrac{3}{5}p && (1) \\ c &= \tfrac{19{,}000}{5} - \tfrac{3(3{,}000)}{5} \\ c &= 2{,}000 \end{aligned}$$

Answer: $c = 2{,}000,\ \ p = 3{,}000$

3. The cost is Le 2,000.00 for a cup and Le 3,000.00 for a plate.

   Solve for a and b from the equations $2^{3b+5a} = 256$ and $7^{9b-5a} = 1$.

   Solution:
   Rewrite the right-hand sides of the equations using the facts that $256 = 2^8$ and $1 = 7^0$. Once the two sides of the equation have indices with the same base, their powers can be set equal to one another:

   $2^{3b+5a} = 256 = 2^8$ $\quad$ $7^{9b-5a} = 1 = 7^0$

   $3b + 5a = 8$ (1) $\quad$ $9b - 5a = 0$ (2)

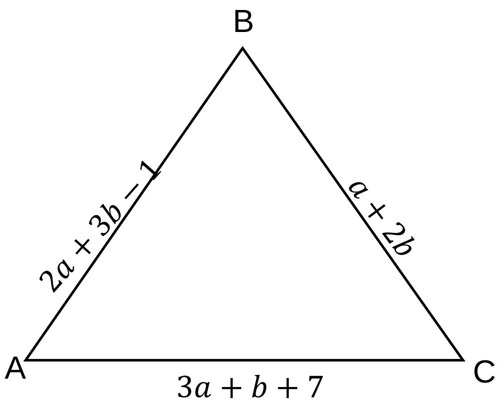
   Solve the system of linear equations:
   Elimination:

$$\begin{aligned} 3b + 5a &= 8 && (1) \\ 9b - 5a &= 0 && (2) \\ 12b &= 8 && \text{Add (1) and (2)} \\ b &= \tfrac{8}{12} = \tfrac{2}{3} && \text{Divide throughout by 12} \\ 3\left(\tfrac{2}{3}\right) + 5a &= 8 && \text{Substitute for } b = \tfrac{2}{3} \text{ in (1)} \\ a &= \tfrac{6}{5} \end{aligned}$$

   Answer: $a = \frac{6}{5}, b = \frac{2}{3}$

Practice

1. The sum of the ages of a father and his son is 46 years. In six years' time, he will be 22 years older than his son. Find their present ages.
2. A woman is four times older than her son, in eight years' time she will be 24 years older. How old are they presently?
3. The sum of the digits of a number is 12. When the digits are interchanged, the number is reduced by 18. What is the number?
4. Four pineapples and two mangoes cost Le 30,000.00, two pineapples and 6 mangoes cost Le 22,000.00. What is the cost of each pineapple and mango?
5. A driver takes 2 hours to travel a certain distance. If he increases his speed by 20km/hr, he takes $1\frac{3}{5}$ hours to cover the same distance. What is the speed and distance he covers? Use the formula $speed = \frac{distance}{time}$.
6. A man is two times as old as his daughter. Eight years ago, he was 15 years older. What will be their ages in 20 years' time?
7. Six apples and 4 tins of milk cost Le 38,000.00. 3 apples and 6 tins of milk cost Le 39,000.00. Find the cost of an apple and a tin of milk.
8. In the figure below, ΔABC is an equilateral triangle. Find the values of a and b.



9. The line $y = mx + c$ passes through the points (1, -5) and (2, 0). Find the values of m and c by substituting each point into the equation to give 2 linear equations.
10. Solve for a and b from the equations $5^{2b+a} = 1$ and $\frac{3^{2a+b}}{27} = 1$

Lesson Title: Simultaneous quadratic and linear equations	**Theme:** Algebra
Practice Activity: PHM4-L030	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve simultaneous quadratic and linear equations using substitution and graphing.

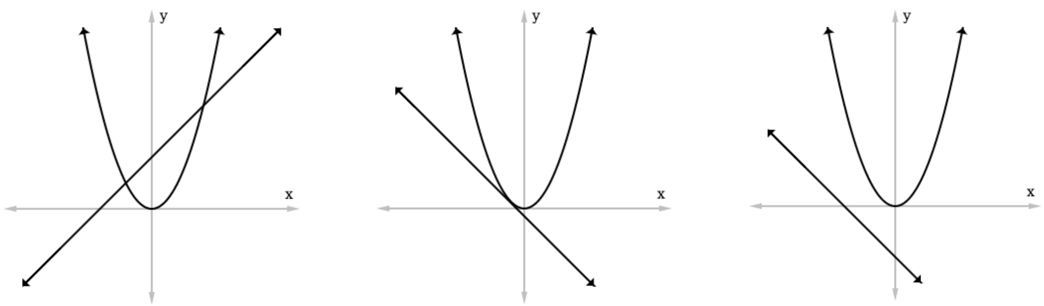
Overview

Simultaneous equations may have a linear and a quadratic equation. The solutions are variables that satisfy both equations. These are simultaneous linear and quadratic equations:

$$y = x + 2 \quad (1) \quad \text{Linear}$$
$$y = x^2 \quad (2) \quad \text{Quadratic}$$

Simultaneous linear and quadratic equations can be solved using substitution or graphing. They can have 0, 1 or 2 solutions. The solutions are ordered pairs, (x, y).

When we graph simultaneous linear and quadratic equations, the intersection points of the curve and line are the solutions to the simultaneous equations.



2 solutions **1 solution** **0 Solutions**

Note that when the simultaneous equations have 1 solution, the line is a tangent touching the parabola at one point. When there are 0 solutions, the line and parabola do not intersect.

To solve simultaneous linear and quadratic equations using **substitution:**

- Solve one equation for x or y, then substitute it into the other equation.
- Simplify until it has the form of a standard quadratic equation ($ax^2 + bx + c = 0$).
- Solve the quadratic equation using any method. Factorisation is shown here.
- Substitute each result into one of the original equations to find the corresponding x- or y-values.

To solve simultaneous linear and quadratic equations by graphing, simply graph both relations on the same axes and observe any points of intersection. The solution is written as an ordered pair (x, y).

Solved Examples

1. Solve**:**

$$\begin{aligned} y &= x^2 - 5x + 7 \qquad (1) \\ y - 2x &= 1 \qquad (2) \end{aligned}$$

Solution:

$(x^2 - 5x + 7) - 2x$	$=$	1	(2)	Substitute equation (1) into equation (2)
$x^2 - 7x + 7$	$=$	1		Simplify
$x^2 - 7x + 7 - 1$	$=$	0		Transpose 1
$x^2 - 7x + 6$	$=$	0		
$(x - 6)(x - 1)$	$=$	0		Factorise the quadratic equation
$x - 6 = 0$	or	$x - 1 = 0$		Set each binomial equal to 0
$x = 6$	or	$x = 1$		Transpose -6 and -1
$y - 2(6)$	$=$	1		Substitute $x = 6$ into equation (2)
$y - 12$	$=$	1		
y	$=$	$1 + 12$		Transpose -12
y	$=$	13		
$y - 2(1)$	$=$	1		Substitute $x = 1$ into equation (2)
$y - 2$	$=$	1		
y	$=$	$1 + 2$		Transpose -2
y	$=$	3		

Answers: $(6, 13)$ and $(1, 3)$

2. Solve:

$$\begin{aligned} 3x + 2y &= 9 \qquad (1) \\ x^2 + y^2 &= 10 \qquad (2) \end{aligned}$$

Solution:

x	$=$	$\frac{9-2y}{3}$	Make x the subject of equation (1)
$\left(\frac{9-2y}{3}\right)^2 + y^2$	$=$	10	Substitute for x in (2)

$$\left(\frac{81-36y+4y^2}{9}\right)+y^2 = 10$$

$81-36y+4y^2+9y^2 = 90$ Multiply throughout by 90

$13y^2-36y-9 = 0$ Transpose 90 and simplify

$(y-3)(13y+3) = 0$ Factorise

$13y+3=0$ or $y-3=0$ Set each binomial equal to 0

$y=-\frac{3}{13}$ or $y=3$ 1

$x = \frac{9-2\left(-\frac{3}{13}\right)}{3}$ Substitute $y=-\frac{3}{13}$ into equation (1)

$x = \frac{41}{13}$ Simplify

$x = \left(\frac{9-2(3)}{3}\right)$ Substitute $y=3$ into equation (1)

$x = 1$ Simplify

Answers: $(1,3)$ and $\left(\frac{41}{13},-\frac{3}{13}\right)$

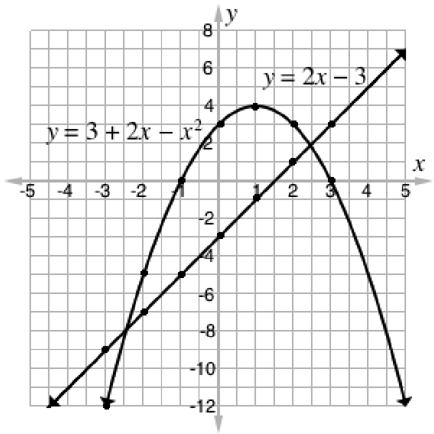
3. Using the scale of 2 cm to 1 unit on the x-axis and 1 cm to 1 unit on the y-axis, draw on the same axes the graphs of $y=3+2x-x^2$ and $y=2x-3$ on the interval $-3\le x\le 3$. Using your graph, find the solutions to the simultaneous equations $y=3+2x-x^2$ and $y=2x-3$.

Solution:

Create a table of values for the graphs on the interval $-3\le x\le 3$.

x	-3	-2	-1	0	1	2	3
$y=3+2x-x^2$	-12	-5	0	3	4	3	0
$y=2x-3$	-9	-7	-5	-3	-1	1	3

Graph the lines on the same axes (graph not to scale):



The parabola and line intersect at $(-2.5,-8)$ and $(2.5,2)$. These are the solutions.

Practice

1. Solve the following equations using substitution:
    a. $y = x^2 + 3x + 1,\ y = -2x - 5$
    b. $y - x^2 = 9,\ 10x + y = 0$
    c. $2l + 3m = 4,\ 9m^2 + 8l = 12$
2. Solve the following simultaneous equations by graphing:
    a. $y = x + 3$ and $y = x^2 - 2x + 3$
    b. $y = -x^2 + x + 2$ and $y + 2 = x$
    c. $y = -x - 4 = 2 - x^2$

Lesson Title: Tangent to a quadratic function	**Theme:** Algebra
Practice Activity: PHM4-L031	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve problems involving the tangent line to a quadratic function.

Overview

This lesson introduces the gradient of a curve. The gradient of a curve changes from point to point. The gradient at any point on a curve is the same as the gradient of the tangent line at that exact point.

To draw a tangent line:

- A tangent line touches the curve at only one point.
- A tangent to a curve at point P can be drawn by placing a straight edge on the curve at P, then drawing a line.
- The "angles" between the curve and line should be nearly equal.
- The parabola and tangent line must be drawn very accurately and clearly to find the correct gradient. Use a very exact scale on the x- and y-axes.

To find the gradient of a curve at point P:

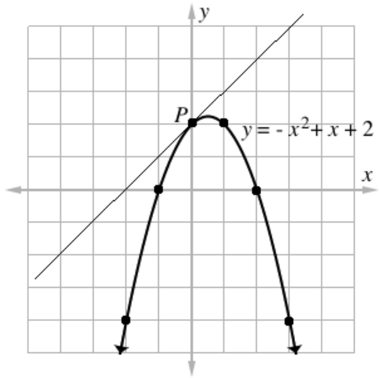
- Draw the tangent at point P.
- Find any 2 points on the tangent line and use them in the gradient formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Solved Examples

1. Graph $y = -x^2 + x + 2$ on the Cartesian plane. Find the gradient at point $P(0, 2)$ by drawing a tangent line.
   Solution:
   Step 1. Graph the parabola by creating a table of values:

x	-2	-1	0	1	2	3
y	-4	0	2	2	0	-4

Step 2. Draw the tangent at point $P(0, 2)$:



Step 3. Identify any two points on the tangent line.

Example: $(x_1, y_1) = (-2, 0)$ and $(x_2, y_2) = (0, 2)$

Step 4. Substitute these coordinates into the gradient formula and solve:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2-0}{0-(-2)} && \text{Substitute } x\text{- and } y\text{-values.} \\ &= \frac{2}{2} && \text{Simplify} \\ m &= 1 \end{aligned}$$

By finding the gradient of the tangent line at point P we have also found that the gradient of the curve $y = -x^2 + x + 2$ at P is $m = 1$.

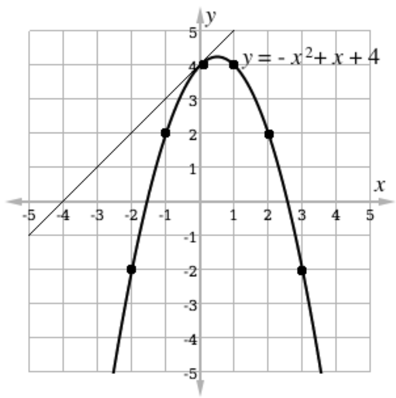
2. Graph $y = 4 + x - x^2$ for $-2 \leq x \leq 3$ on the Cartesian plane. Find the gradient at $x = 0$.

Solution:

Step 1. Graph the parabola by creating a table of values:

x	-2	-1	0	1	2	3
y	-2	2	4	4	2	-2

Step 2. Draw the tangent at $x = 0$:



Step 3. Identify two points on the tangent line.
$(x_1, y_1) = (-1, 3)$ and $(x_2, y_2) = (0, 4)$.

Step 4. Substitute these coordinates into the gradient formula and solve:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{4-3}{0-(-1)} && \text{Substitute } x\text{- and } y\text{-values.} \\ &= \frac{1}{1} && \text{Simplify} \\ m &= 1 \end{aligned}$$

By finding the gradient of the tangent line at $x = 0$, the gradient of the curve $y = 4 + x - x^2$ at $x = 0$ is also $m = 1$.

3. a. Complete the table of values for the relation $y = x^2 + x - 2$.

x	-3	-2	-1	0	1	2
y						

b. Draw the graph of y for $-3 \leq x \leq 2$.

c. From your graph:

i. Find the roots of the equation $x^2 + x - 2 = 0$.

ii. Estimate the minimum value of y.

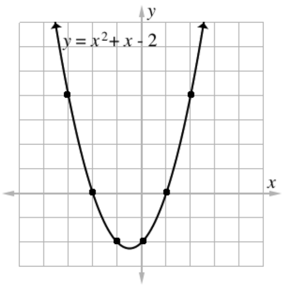
iii. Calculate the gradient of the curve at the point $x = 0$.

Solutions:

a.

x	-3	-2	-1	0	1	2
y	4	0	-2	-2	0	4

b.

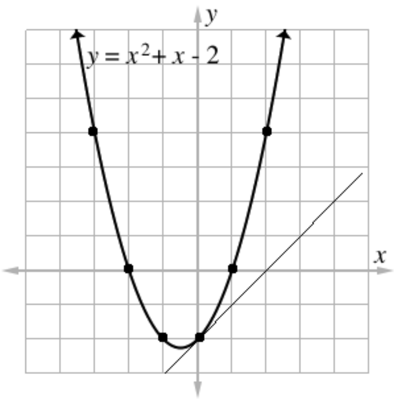


c.

i. The roots are the x-intercepts: $x = -2$ and $x = 1$

ii. The minimum value of y is slightly less than -2. Estimate: $y = -2.2$

iii. Draw the tangent line at $x = 0$, and find its gradient:



Identify any two points on the tangent. Example: $(x_1, y_1) = (0, -2)$ and $(x_2, y_2) = (2, 0)$). Use them to find the gradient of the curve:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{0-(-2)}{2-0}$$ Substitute x- and y-values.

$$= \frac{2}{2}$$ Simplify

$$m = 1$$

4. a. Complete the following table of values for the relation: $y = 2x^2 - 7x - 3$

x	-2	-1	0	1	2	3	4	5
y	19		-3		-9			

b. Using 2 cm to 1 unit on the x-axis and 2 cm to 5 units on the y-axis, draw the graph of $y = 2x^2 - 7x - 3$.

c. From your graph find the following:

i. Minimum value of y.

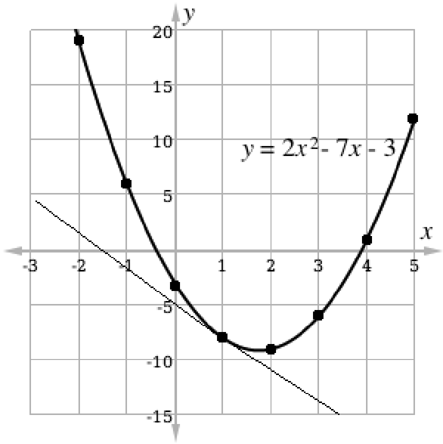
ii. Gradient of the curve at $x = 1$.

Solutions:

a. Table:

x	-2	-1	0	1	2	3	4	5
y	19	6	-3	-8	-9	-6	1	12

b. Draw the graph. Note that the graph below is not printed to scale. The marks on the x- and y-axes should be 2 cm apart in your graph.



c. i. Minimum value of $y = -9$

ii. Choose 2 points on the tangent line (Example: $(0, -5)$ and $(1, -8)$). Calculate the gradient of the curve at $x = 1$:

$$m = \frac{-8-(-5)}{1-0}$$
$$= \frac{-8+5}{1}$$
$$m = -3$$

Practice

1. Draw the graph of $y = x^2 - 2x + 1$ on the Cartesian plane. Find gradient of the curve at $x = 3$.
2. Graph $y = -2x^2 + 5x + 1$, and estimate the gradient of the curve at $x = 2$.
3. Draw the graph of $y = x^2 + x - 3$ for $-3 \leq x \leq 2$ on the Cartesian plane. Find the gradient of the curve at $x = -1$.
4. Complete the following for the relation $y = -x^2 + x + 6$
   a. Complete the table of values:

x	-2	-1	0	1	2	3
y						

   b. Draw the graph of the relation.
   c. Draw the tangent line at $x = 1$ and find the gradient of the curve.

Lesson Title: Inequalities	**Theme:** Algebra
Practice Activity: PHM4-L032	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve inequalities in one variable and represent them on number lines.

Overview

Inequalities in one variable are any inequalities that include a variable, inequality symbol (<, >, ≥ or ≤), and one or more numbers.

For example, these are inequalities in one variable:

$x \geq -6$ $\qquad$ $y < -6$ $\qquad$ $x + 3 > 4$ $\qquad$ $4 \leq \frac{b-9}{3}$

Inequalities can be solved using nearly the same process that is used for equations. Here are some steps you can take to solve an inequality:

- Add or subtract a term from both sides
- Multiply or divide both sides by a positive number
- Multiply or divide both sides by a negative number (the direction of the inequality is **reversed**)

As with solving an equation, we want to get the variable alone on one side of the inequality. Add or subtract numbers from both sides first, then multiply or divide.

Once you solve an inequality in one variable, the answer will still be in the form of an inequality. The variable will be on one side of the symbol, and a number on the other side. This inequality can be shown on a number line.

Inequalities in one variable can be shown on a number line using circles and arrows. These are the arrows that show each inequality symbol:

Less than (<) $\qquad$ Greater than (>) $\qquad$ Less than or equal to (≤) $\qquad$ Greater than or equal to (≥)

For expressions with two inequalities (example: $-4 \leq x < 2$, there is no arrow. The two end points are shown with circles. For example:

Solved Examples

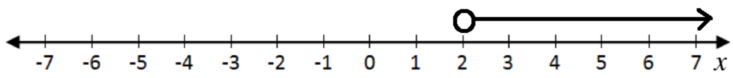
1. Solve the inequality and illustrate the result on a number line: $x + 10 > 12$.

   Solution:

   Step 1. Solve for x:

$$x + 10 > 12$$
$$x > 12 - 10 \quad \text{Transpose}$$
$$x > 2$$

   Step 2. Draw a number line with an open circle at 2, and an arrow to the right.



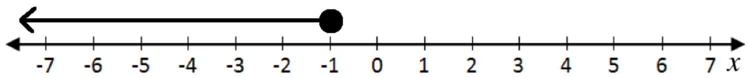
2. Solve the inequality and illustrate the result on a number line: $x + 8 \leq 7$

   Solution:

   Step 1. Solve for x:

$$x + 8 \leq 7$$
$$x \leq 7 - 8 \quad \text{Transpose 8}$$
$$x \leq -1$$

   Step 2. Draw a number line with a filled circle at $x = -1$, and an arrow to the left:



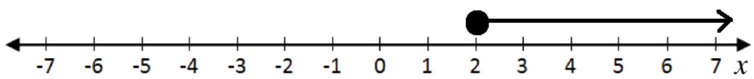
3. What is the range of values of x which satisfies $\frac{1}{3}(2x - 1) - \frac{1}{4}(x - 2) \geq 1$? Display the answer on a number line.

   Solution:

   Step 1. Solve for x:

$$\frac{1}{3}(2x - 1) - \frac{1}{4}(x - 2) \geq 1$$
$$12\times\frac{1}{3}(2x - 1) - 12\times\frac{1}{4}(x - 2) \geq 12 \quad \text{Multiply throughout by the LCM, 12}$$
$$4(2x - 1) - 3(x - 2) \geq 12$$
$$8x - 4 - 3x + 6 \geq 12 \quad \text{Clear the brackets}$$
$$5x \geq 12 - 6 + 4 \quad \text{Collect like terms}$$
$$5x \geq 10$$
$$x \geq 2 \quad \text{Divide throughout by 2.}$$

   Step 2. Draw a number line with a filled circle at $x = 2$, and an arrow to the right.



4. Write $-13 < 5x - 3 \leq 2$ in the form $a < x \leq b$, where a and b are integers, and represent the solution on the number line.

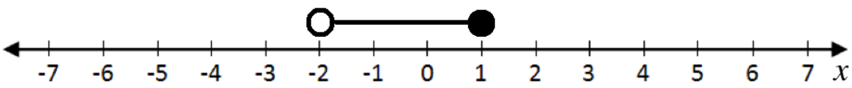
   Solution:

Step 1. Solve for x:

$$\begin{aligned} -13+3 &< 5x-3+3 \leq 2+3 && \text{Add 3 throughout} \\ -10 &< 5x \leq 5 \\ \frac{-10}{5} &< \frac{5x}{5} \leq \frac{5}{5} && \text{Divide throughout by 5} \\ -2 &< x \leq 1 \end{aligned}$$

$-2 < x \leq 1$ can also be written as 2 inequalities: $x > -2$ and $x \leq 1$

Step 2. Draw a number line with a filled circle at $x = 1$, and an arrow to the left; and an open circle at -2, and an arrow to the right:



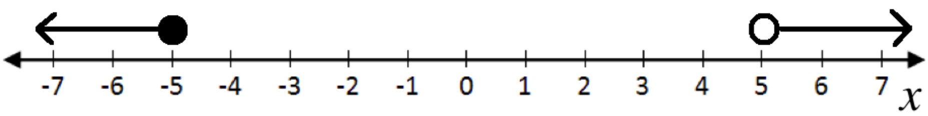
5. Solve: $5x + 17 \leq 2x + 2$ or $6x + 5 > 4x + 15$ and represent the solution on the number line.

Solution:

Step 1. Solve for x:

$$\begin{aligned} 5x+17 &\leq 2x+2 && \text{Or} && 6x+5 > 4x+15 \\ 5x-2x &\leq 2-17 && \text{Or} && 6x-4x > 15-5 \\ 3x &\leq -15 && \text{Or} && 2x > 10 \\ x &\leq -5 && \text{Or} && x > 5 \end{aligned}$$

Step 2. Draw a number line with a filled circle at $x = -5$, and an arrow to the left; and an open circle at 5, and an arrow to the right.



Practice

1. Find the range of values of y for which $\frac{1}{3}(y+3) - 2(y-5) > 4\frac{1}{3}$. Display the answer on the number line.
2. If $4 + x \leq 7$ and $9 + x \geq 5$, what range of values of x satisfies both inequalities? Represent the solution on the number line.
3. What is the range of values of x for which $3x + 1 > 10$ and $2x - 5 \leq 9$ are both satisfied? Represent the solution on the number line.
4. What is the range of values of a for which $5(1 - a) > 5$ and $7(1 + a) \geq 0$? Represent the solution on the number line.
5. Express $8x - 6 \leq 6x \leq 8x + 10$ in the form $a \leq x \leq b$ where a and b are integers and represent the solution on the number line.
6. Express $5 - y \leq 7 < 12 - y$ in the form $a \leq y < b$ where a and b are integers.
7. Solve and represent the solution on the number line: $6x + 3 < 4x + 1$ or $7x + 11 \geq 2x + 21$

Lesson Title: Variation	Theme: Algebra
Practice Activity: PHM4-L033	**Class:** SSS 4

Learning Outcomes

By the end of the lesson, you will be able to:

1. Identify and differentiate between direct, indirect, joint, and partial variation.
2. Solve variation problems.

Overview

This lesson covers 4 types of variation: direct, indirect, joint, and partial. Each is described below.

Direct Variation:

Direct variation means that two quantities x and y are related such that an increase in one results in an increase in the other in the same ratio. At the same time, a decrease in one results in a decrease in the other in the same ratio. We can also say that x and y are "directly proportional".

Direct variation can be shown with the symbol $\propto$. The statement $x \propto y$ means that x and y are directly proportional.

Direct variation can be shown with the equation $x = ky$, where k is a constant, and x and y represent two quantities that are directly proportional.

Indirect Variation:

Indirect variation means that two quantities x and y are related such that an increase in one results in a decrease in the other. At the same time, a decrease in one results in an increase in the other. We can also say that x and y are "indirectly proportional".

Indirect variation uses the same symbol as direct variation: $\propto$. The statement $x \propto \frac{1}{y}$ means that x and y are indirectly proportional.

Indirect variation can be shown with the equation $x = k\frac{1}{y}$ or $x = \frac{k}{y}$, where k is a constant, and x and y represent two quantities that are indirectly proportional.

Joint Variation:

Joint variation occurs when a variable varies directly or inversely with multiple variables. For example:

- If x varies directly with both y and z, we have $x \propto yz$ or $x = kyz$.
- If x varies directly with y and inversely with z, we have $x \propto \frac{y}{z}$ or $x = \frac{ky}{z}$.

Joint variation problems can be solved using the same process as other variation problems: Set up the equation and substitute the given set of 3 values to find the constant k. Write the equation with constant k, and substitute the other 2 given values to find the answer.

Partial Variation:

Partial variation occurs when a variable is related to two or more other variables added together. This lesson focuses on the case where a variable (such as x) is partly a constant, and partly varies directly with another variable (such as y).

For example, $x = k_1 + ky$ states that x is partly related to the constant k_1, and varies partly as y.

In many cases, you will be asked to determine the relationship between two variables (such as x and y) that are related by partial variation. This involves finding two constants, k_1 and k. These can be found by forming two simultaneous linear equations.

Solved Examples

1. x varies inversely as the square of $(y - 3)$. Given $x = 1$, when $y = 5$, find:
   a. The relation between x and y
   b. The possible values of y when $x = 64$

Solutions:

a.

x	$\propto$	$\frac{1}{(y-3)^2}$	Identify the relationship between x and y
x	$=$	$\frac{k}{(y-3)^2}$	Write as a formula with constant k
1	$=$	$\frac{k}{(5-3)^2}$	Substitute known values for x and y
k	$=$	2^2	Solve for the constant, k
k	$=$	4	
x	$=$	$\frac{4}{(y-4)^2}$	This is the relation between x and y

b.

x	$=$	$\frac{4}{(y-3)^2}$	Formula
64	$=$	$\frac{4}{(y-3)^2}$	Substitute $x = 64$
$(y-3)^2$	$=$	$\frac{4}{64}$	
$(y-3)$	$=$	$\sqrt{\frac{4}{64}}$	
$y-3$	$=$	$\pm\frac{1}{4}$	
y	$=$	$3 \pm \frac{1}{4}$	Solve for y

Answers: $y = 3\frac{1}{4}$ or $y = 2\frac{3}{4}$

2. The cost C of tiling a rectangular wall is partly constant and partly varies jointly as the length, L, and height H, of the wall. The cost of tiling a wall of length 5 m and height 2 m is Le 500,000.00 and for a wall of length 8 m and height 3 m, the cost is Le 780,000.00. Find:
    a. The relation between C, L and H.
    b. The cost of tiling a wall of length 12 m and height 5 m.

Solutions:
Write the relationship with variables: $C = k_1 + kLH$

a. Substitute the given values of C, L and H into $C = k_1 + kLH$, which gives two equations:
$500{,}000 = k_1 + (5)(2)k;\ \ 500{,}000 = k_1 + 10k$-----------------(1)

$780{,}000 = k_1 + (8)(3)k;\ \ 780{,}000 = k_1 + 24k$-----------------(2)

Solve the simultaneous equations by subtracting (1) from (2):

$780{,}000$	$=$	$k_1 + 24k$	(2)	
$-(500{,}000$	$=$	$k_1 + 10k)$	(1)	Subtract each term
$280{,}000$	$=$	$14k$		
$\frac{280{,}000}{14}$	$=$	$\frac{14k}{14}$		Divide throughout by 14
$20{,}000$	$=$	k		
$500{,}000$	$=$	$k_1 + 10(20{,}000)k$	(1)	Substitute $k = 20{,}000$ into (1)
$500{,}000$	$=$	$k_1 + 20{,}0000$		Simplify
$500{,}000 - 200{,}000$	$=$	k_1		Transpose 200,000
$300{,}000$	$=$	k_1		

Answer: The relationship is $C = 300{,}000 + 20{,}000LH$

b. Substitute the given values into the formula:

C	$=$	$300{,}000 + 20{,}000LH$	Relationship
C	$=$	$300{,}000 + 20{,}000(12)(5)$	Substitute $L = 12; H = 5$
C	$=$	$300{,}000 + 1{,}200{,}000$	Simplify
C	$=$	$1{,}500{,}000$	

Answer: The cost of painting a wall of length 12 m and height 5 m is Le 1,500,000.00

Practice

1. T varies directly as U and the square of $(V - 4)$. Given $T = 32, U = 2\ and\ V = 6$, find:
    a. The relation between $T, U\ and\ V$.

b. The possible values of V when $T = 128$ and $U = 8$.

2. The pressure, P, on a surface varies directly as the force, F, exerted and inversely as the surface area, A, of the surface. For a force of 60N on an area of 20 m^2, the pressure is 6 N/m^2. Find:
   a. An expression relating P, F and A.
   b. The force which exerts a pressure of 150 N/m^2 on an area of 5 m^2.
3. The cost C of painting a house is partly constant and partly varies as the number of rooms n, in the house. The cost for painting a house with 5 rooms is $\text{Le } 700{,}000.00$ and for a house with 7 rooms the cost is $\text{Le } 900{,}000.00$. Find:
   a. An expression for C in terms of n.
   b. The cost of painting a house with 12 rooms.
   c. The number of rooms in a house that cost $\text{Le } 1{,}600{,}000.00$ for painting.
4. The electrical resistance, R, of a wire varies directly as the length, l, and inversely as the square of the radius,r, of the wire. The resistance is $6\ \Omega$ (ohms) when the length is 12 cm and radius 0.2 cm. Find:
   a. The relation between $R, l \; and \; r$.
   b. The resistance when the length is 50 cm and radius 0.25 cm.
5. The force of attraction, F, between two bodies varies directly as the product of their masses m_1 and m_2 and inversely as the square of the distance, s, between them. Given that $F = 25N$ when $m_1 = 25 \text{ kg}$, $m_2 = 75 \text{ kg}$ and $s = 15 \text{ m}$. Find:
   a. An expression for F in terms of m_1, m_2 and s.
   b. The mass, m_2 when $F = 45N, m_1 = 1.5 \text{ kg}$ and $s = 3 \text{ m}$
6. The resistance, R to the motion of a car is partly constant and partly varies as the square of the speed U, of the car. When the car is moving at 20 km/h, the resistance is $350N$ and when it is moving at 25 km/h, the resistance is $1{,}025N$. Find:
   a. An expression for R in terms of U.
   b. The resistance at 50 km/h.

Lesson Title: Simplification of algebraic fractions	**Theme:** Algebra
Practice Activity: PHM4-L034	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to use factorisation to simplify algebraic fractions.

Overview

Algebraic fractions are fractions with variables in them. These are the steps for simplifying algebraic fractions:

1. Factor the numerator and denominator if possible.
2. Find the highest common factor (HCF) of the numerator and denominator. Remember that the HCF is simply all of the common factors multiplied together.
3. Divide (cancel) the HCF from the numerator and denominator.

Solved Examples

1. Simplify: $\frac{6x^2yz^2}{12xy^2z}$.
   Solution:
   Identify that the HCF of the numerator and denominator is $6xyz$.

   Divide the numerator and denominator by $6xyz$: $\frac{6x^2yz^2 \div 6xyz}{12xy^2z \div 6xyz} = \frac{xz}{2y}$

2. Reduce $\frac{3x+6y}{x^2+2xy}$ to its lowest terms.
   Solution:
   Step 1. Factor the numerator and denominator:
   $$\frac{3x+6y}{x^2+2xy} = \frac{3(x+2y)}{x(x+2y)}$$
   Step 2. Identify that the HCF in the numerator and denominator is $(x+2y)$.
   Step 3. Divide (cancel) the numerator and denominator by $(x+2y)$:
   $$\frac{3(x+2y)}{x(x+2y)} = \frac{3}{x}$$

3. Simplify: $\frac{x^2-x-6}{x^2+2x}$
   Solution:
   Step 1. Factor the numerator and denominator:
   $$\frac{x^2-x-6}{x^2+2x} = \frac{(x-3)(x+2)}{x(x+2)}$$
   Step 2. Identify that the HCF in the numerator and denominator is $(x+2)$.
   Step 3. Divide (cancel) the numerator and denominator by $(x+2)$:

$$\frac{(x-3)(x+2)}{x(x+2)} = \frac{x-3}{x}$$

4. Simplify: $\frac{2x^2+5x+3}{x^2-2x-3}$

Solution:

Step 1. Factor the numerator and denominator:

$$\frac{2x^2+5x+3}{x^2-2x-3} = \frac{(2x+3)(x+1)}{(x-3)(x+1)}$$

Step 2. Identify that the HCF in the numerator and denominator is $(x+1)$.

Step 3. Divide (cancel) the numerator and denominator by $(x+1)$:

$$\frac{(2x+3(x+1)}{(x-3)(x+1)} = \frac{2x+3}{x-3}$$

5. Simplify: $\frac{1-m^2}{3m^2-m-2}$

Solution:

Step 1. Factor the numerator and denominator:

$$\frac{1-m^2}{(3m^2-m-2)} = \frac{(1-m)(1+m)}{(m-1)(3m+2)}$$

There are no common factors in the numerator and denominator. However, we can factor -1 from $(m-1)$ in the denominator to get $(-m+1) = (1-m)$, which is a common factor with the numerator. Therefore, we have:

$$\frac{1-m^2}{(3m^2-m-2)} = \frac{(1-m)(1+m)}{(m-1)(3m+2)} = \frac{(1-m)(1+m)}{-(1-m)(3m+2)}$$

Step 2. Identify that the HCF in the numerator and denominator is $(1-m)$.

Step 3. Divide (cancel) the numerator and denominator by $(1-m)$:

$$-\frac{(1-m)(1+m)}{(1-m)(3m+2)} = -\frac{1+m}{3m+2}$$

Practice

1. Reduce to its lowest terms: $\frac{18ab^2c^2}{24a^2bc}$
2. Simplify: $\frac{2x+10y}{x^2+5xy}$
3. Simplify: $\frac{x^2-5x-6}{x^2-6x-7}$
4. Simplify: $\frac{3x^2-x-2}{x^2+2x-3}$
5. Simplify: $\frac{1-2m-3m^2}{3m^2+5m-2}$
6. Simplify: $\frac{3m^2-7m+2}{2+5m-3m^2}$

Lesson Title: Operations on algebraic fractions	**Theme:** Algebra
Practice Activity: PHM4-L035	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to apply basic operations (addition, subtraction, multiplication, division) to algebraic fractions and reduce them to their lowest terms.

Overview

These are the steps for **multiplying** algebraic fractions:

1. Factor the numerators and denominators.
2. Multiply the numerators and denominators. This is the same step we take when multiplying fractions with numbers.
3. Identify the common factors in the numerator and denominator.
4. Divide the numerators and denominators by the factors they have in common. In other words, cancel factors that are in both the numerator and denominator.
5. Leave the result as your answer. If there are brackets, you do not need to multiply them out.

These are the steps for **dividing** algebraic fractions:

1. Multiply by the reciprocal (inverse) of the second fraction. This is the same step we take when dividing fractions with numbers.
2. Follow the steps for multiplication of algebraic fractions (above).

These are the steps for **adding** or **subtracting** algebraic fractions:

1. Factor the algebraic fractions if possible.
2. To add or subtract algebraic fractions, the denominators should be the same. Express each fraction with a denominator that is the LCM of the denominators in the problem.
3. Once the fractions have the same denominator, they are like fractions and can be added or subtracted.
4. Combine like terms if possible.
5. If the answer can be simplified, simplify it. This requires factoring the answer if possible.

Solved Examples

1. Write $\frac{3}{2(x-1)} - \frac{3}{2(x+1)}$ as a single fraction.
   Solution:
   Note that the LCM of the denominators is $2(x-1)(x+1)$.

$\frac{3}{2(x-1)} - \frac{3}{2(x+1)}$	$= \frac{3(x+1)-3(x-1)}{2(x-1)(x+1)}$	Divide the LCM by the denominator of each fraction and multiply by the numerator of that fraction.
	$= \frac{3x+3-3x+3}{2(x-1)(x+1)}$	Clear the brackets of the numerator
	$= \frac{6}{2(x-1)(x+1)}$	Combine like terms
	$= \frac{3}{(x-1)(x+1)}$	Divide the numerator and denominator by 2

2. Simplify: $1 - \frac{7}{4(x+3)} - \frac{1}{4(x-1)}$

Solution:

Note that the LCM of the denominators is $4(x+3)(x-1)$

$1 - \frac{7}{4(x+3)} - \frac{1}{4(x-1)}$	$= \frac{4(x+3)(x-1)-7(x-1)-(x+3)}{4(x+3)(x-1)}$	Divide the LCM by the denominator of each fraction and multiply
	$= \frac{4(x^2+2x-3)-7x+7-x-3}{4(x+3)(x-1)}$	Clear the brackets of the numerator
	$= \frac{4x^2+8x-12-7x+7-x-3}{4(x+3)(x-1)}$	
	$= \frac{4x^2-8}{4(x+3)(x-1)}$	Combine like terms
	$= \frac{4(x^2-2)}{4(x+3)(x-1)}$	Factorise the numerator
	$= \frac{x^2-2}{(x+3)(x-1)}$	Divide the numerator and denominator by 4

3. Simplify: $\frac{2x+1}{x^2+3x+2} + \frac{2}{x+2}$

Solution:

Note that the denominators are not factored. They must be factored first.

Step 1. Factorise the denominator:

$$\frac{2x+1}{x^2+3x+2} + \frac{2}{x+2} = \frac{(2x+1)}{(x+2)(x+1)} + \frac{2}{x+2}$$

Step 2. Find the LCM of the denominators: Find the LCM of the denominators: $(x+2)(x+1)$

Step 3. Add the algebraic fractions:

$\frac{(2x+1)}{(x+2)(x+1)} + \frac{2}{x+2}$	$= \frac{(2x+1)+2(x+1)}{(x+2)(x+1)}$	Divide the LCM by the denominator of each fraction and multiply by the numerator of that fraction.
	$= \frac{2x+1+2x+2}{(x+2)(x+1)}$	Clear the brackets of the numerator
	$= \frac{4x+3}{(x+2)(x+1)}$	Combine like terms

4. Simplify: $\frac{x^2-16}{x^2+3x} \times \frac{x+3}{x^2+5x+4}$

Solution:

$\frac{x^2-16}{x^2+3x} \times \frac{x+3}{x^2+5x+4}$	$=$	$\frac{(x+4)(x-4)}{x(x+3)} \times \frac{(x+3)}{(x+4)(x+1)}$	Factorise the denominators and the numerators
	$=$	$\frac{(x+4)(x-4)(x+3)}{x(x+3)(x+4)(x+1)}$	Multiply numerators and denominators
	$=$	$\frac{x-4}{x(x+1)}$	The factors that cancelled out are $(x+4)$ and $(x+3)$

5. Simplify: $\frac{x^2-9}{5x^2+10x} \div \frac{x^2-x-12}{x^2+2x}$

Solution:

$\frac{x^2-9}{5x^2+10x} \div \frac{x^2-x-12}{x^2+2x}$	$=$	$\frac{(x+3)(x-3)}{5x(x+2)} \div \frac{(x+3)(x-4)}{x(x+2)}$	Factorise the numerators and denominators
	$=$	$\frac{(x+3)(x-3)}{5x(x+2)} \times \frac{x(x+2)}{(x+3)(x-4)}$	Multiply by the reciprocal fraction
	$=$	$\frac{x(x+3)(x-3)(x+2)}{5x(x+2)(x+3)(x-4)}$	
	$=$	$\frac{x-3}{5(x-4)}$	The factors that cancelled out are $(x+3), x,$ and $(x+2)$

6. If $m = \frac{1}{2x}$ and $n = \frac{1}{1+x}$, express $m-2n$ in terms of x.

Solution:

Substitute the given fractions, and simplify.

$m-2n$	$=$	$\frac{1}{2x} - 2\left(\frac{1}{1+x}\right)$	Substitute
	$=$	$\frac{1}{2x} - \frac{2}{1+x}$	Simplify
	$=$	$\frac{(1+x)-2(2x)}{2x(1+x)}$	Subtract
	$=$	$\frac{1+x-4x}{2x(1+x)}$	Remove brackets in the numerator
	$=$	$\frac{1-3x}{2x(1+x)}$	Combine like terms

Practice

1. Write as a single fraction: $\frac{3}{4(x+2)} + \frac{1}{4(x-2)}$.
2. Simplify: $1 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$
3. If $\frac{4}{2y-5} - \frac{5}{y+4}$ is equal to $\frac{m}{(2y-5)(y+4)}$, find m.
4. Write as a single fraction: $1 + \frac{1}{5(x-1)} - \frac{1}{2(x+1)}$
5. Simplify: $\frac{x^2-49}{4x^2+12x} \times \frac{x^2+2x-3}{x-7}$

6. Simplify: $\frac{y^2-25}{y^2+4y} \times \frac{y+4}{y^2+2y-15}$
7. Simplify: $\frac{a^2+4a-5}{a^2+2a-3} \div \frac{a^2-25}{6a^2+18a}$
8. If, $m = \frac{2x}{1-x}$ and $n = \frac{2x}{1+x}$, express $2m - n$ in terms of x.
9. Simplify: $\frac{\frac{1}{2a}+\frac{1}{3b}}{2a+3b}$

Lesson Title: Logical reasoning – Part 1	Theme: Logical Reasoning
Practice Activity: PHM4-L036	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Distinguish between simple and compound statements.
2. Identify the negation of a statement.
3. Draw conclusions from a given implication.
4. Distinguish between conjunction and disjunction, and represent them on truth tables.
5. Recognise equivalent statements and apply them to arguments.

Overview

The **negation** of a statement has an opposite meaning to the original statement. The negation of a statement is formed by adding the word "not". The negation of a statement p is written as $\sim p$. For example, if p: Francis is a tailor, then $\sim p$: Francis is **not** a tailor.

Compound statements are made from a simple statement and connecting words. Common connecting words are: and, but, or, if and only if, if…then.

Consider the compound statement "Sami is a cat and he likes to eat fish". It has two parts: "Sami is a cat" and "he likes to eat fish", which are connected by the word "and".

Implications are compound statements that can be written with the connecting words "if…then". The first statement implies that the second is true.

Consider the implication, "If Hawa lives in Freetown, then she lives in Sierra Leone." There are 2 simple statements in this implication:

A: Hawa lives in Freetown.
B: She lives in Sierra Leone.

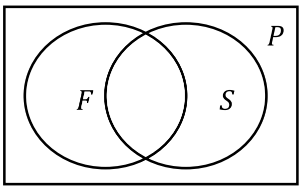
This implication can be written with symbols as $A \Rightarrow B$. Note that if A is true (Hawa lives in Freetown) then B is definitely true. She must live in Sierra Leone because that's where Freetown is located.

A **conjunction** is a compound statement that uses "and". A **disjunction** is a compound statement that uses "or".

Consider 3 sets of people:

P: People who live in Freetown
F: Females
S: Students

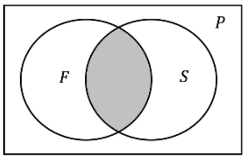
These can be visualised with a Venn diagram:



Consider the compound statements, A_1 and A_2:

A_1: people who are female and are students
A_2: people who are either female, or are students, or both

A_1 is a **conjunction**. The two statements "people who are female" and "people who are students" are linked by the word **"and"**. In the Venn diagram, this is where the two circles intersect.



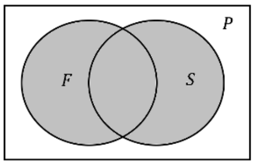
The conjunction can be written in symbols: $A_1 = F \cap S$ or $A_1 = F \wedge S$

A_1 can be represented in a truth table, shown to the right:

F	S	$F \wedge S$
T	T	T
T	F	F
F	T	F
F	F	F

This truth table says that statement A_1 is only true if both of the sub-statements are true ("people who are female" and "people who are students"). If either of the sub-statements is false, then statement A_1 is also false.

A_2 is a **disjunction**. The two statements "people who are female" and "people who are students" are linked by the words **"either – or – or both"**. In the diagram, this is all of the space inside of the two circles, including where they intersect.



The disjunction can be written in symbols: $A_2 = F \cup S$ or $A_2 = F \vee S$

A_2 can be represented in a truth table, shown to the right:

F	S	$F \vee S$
T	T	T
T	F	T
F	T	T
F	F	F

This truth table says that A_2 is true if either or both of the sub-statements are true. A_2 is false only if both sub-statements are false.

Equivalence:

- If $X \Rightarrow Y$ and $Y \Rightarrow X$, then X and Y are equivalent and we write $X \Leftrightarrow Y$. The implication $Y \Rightarrow X$ is the converse of $X \Rightarrow Y$.
  - For example, consider $x^2 = 16 \Rightarrow x = \pm 4$. Its converse is $x = \pm 4 \Rightarrow x^2 = 16$ is true, and we can also write $x^2 = 16 \Leftrightarrow x = \pm 4$.
- It is important to note that if $X \Rightarrow Y$, then $\sim Y \Rightarrow \sim X$ is an equivalent statement.
  - For example, consider the implication: "If Fatu lives in Freetown, then she lives in Sierra Leone." This is an equivalent statement: "If Fatu does not live in Sierra Leone, then she does not live in Freetown."

Solved Examples

1. Consider the following statements:
   S: A shape has 3 sides.
   T: It is a triangle.

   If $S \Rightarrow T$, write the implication with words.

Solution:
If a shape has 3 sides, then it is a triangle.

2. Consider the following statements about some people living in a village:
   C: Some people are children.
   M: Some people are male.

   Based on this information, prepare truth tables that describe:

   a. A person in the village who is a male child.
   b. A person in the village who is either a child, a male, or both.

   Solutions:

   a.

C	M	$C \wedge M$
T	T	T
T	F	F
F	T	F
F	F	F

   b.

C	M	$C \vee M$
T	T	T
T	F	T
F	T	T
F	F	F

3. Consider the following statement: S: The teacher entered and took out her book. Use appropriate symbols and truth tables to describe the conditions for S to be true.
   Solution:
   Determine two sub-statements of S:

   E: The teacher entered.
   B: The teacher took out her book.

   If S is true, then both sub-statements must be true. In other words, $E \wedge B$ must be true. If one of the sub-statements is false, then S will be false.

   Symbols: $S = E \wedge B$; Truth table:

E	B	$E \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

4. Write the converse of the following statements:
   a. If a shape has 3 sides, then it is a triangle.
   b. If you are living, then your heart is beating.
   c. If it is a cat, then it meows.

   Solutions:
   a. If a shape is a triangle, then it has 3 sides.
   b. If your heart is beating, then you are living.
   c. If it meows, then it is a cat.

5. Identify whether the following sets of statements are equivalent:

a. A: $x = \pm 3$ B: $x^2 = 9$
b. S: $x = 4$ T: $x = 3$
c. X: $x = 5$ Y: $x^2 = 25$

Solutions:

a. The statements are equivalent, because $A \Rightarrow B$ and $B \Rightarrow A$.
b. The statements are not equivalent. Neither implication that we can write is true.
c. The statements are not equivalent. Although $X \Rightarrow Y$, it is **not true** to say that $Y \Rightarrow Z$. If we know that $x^2 = 25$, then the implication is that $x = \pm 5$, not $x = 5$.

Practice

1. Connect the following statements in implications. Write the implications using words and symbols.
   a. E: John has measles. F: John is in the hospital.
   b. A: Amid plays football. B: Amid scores many goals.
2. Consider the following statements:
   Y: David studied very hard and passed his Mathematics exam.
   Use appropriate symbols and truth tables to describe the conditions for Y to be true.
3. Consider the following statements about some students in Freetown.
   A: Some of the students are in the science faculty.
   B: Some of the students are members of the science society.
   Based on this information, prepare a truth table that describes:
   a. A student who is either in the science faculty or a member of the science society.
   b. A student who is both in the science faculty and a member of the science society.
4. Consider the true statements below, and write other true statements with this information:
   a. If Foday harvests his farm, then he has a lot to eat.
   b. If Sia lives in Bo, then she lives in Sierra Leone.
   c. If Issa is from Sierra Leone, then he is African.
5. Identify whether the following sets of statements are equivalent:
   a. A: $x = \pm 10$ B: $x^2 = 100$
   b. S: A shape is a square. T: A shape has 4 sides.
   c. X: A shape is a pentagon. Y: A shape has 5 sides.

Lesson Title: Logical reasoning – Part 2	**Theme:** Logical Reasoning
Practice Activity: PHM4-L037	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Apply the chain rule.
2. Use Venn diagrams to demonstrate connections between statements.
3. Determine the validity of statements.

Overview

This lesson focuses on two tools in applying logic: the chain rule and Venn diagrams. You will be using these tools and the information from the previous lesson to solve problems and determine the validity of statements.

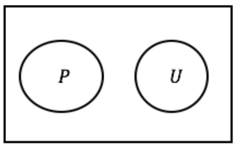
Chain rule:

- If X, Y and Z are 3 statements such that $X \Rightarrow Y$ and $Y \Rightarrow Z$, then $X \Rightarrow Z$.
  - For example, consider the statements, where $X \Rightarrow Y$ and $Y \Rightarrow Z$:
    X: Hawa studies hard.
    Y: Hawa passes exams.
    Z: Hawa graduates secondary school.
  - If $X \Rightarrow Y$ and $Y \Rightarrow Z$, then $X \Rightarrow Z$. In other words, these statements are true:
    $X \Rightarrow Y$: If Hawa studies hard then she passes exams.
    $Y \Rightarrow Z$: If Hawa passes exams then she graduates secondary school.
    $X \Rightarrow Z$: If Hawa studies hard then she graduates secondary school.

Venn Diagrams:

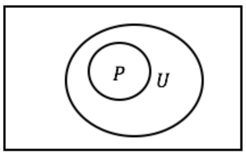
Recall how relationships between sets are shown with Venn diagrams:

X:



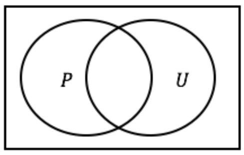
$P \cap U = \emptyset$
P and U are disjoint

Y:



$P \subset U$
P is a subset of U

Z:



$P \cap U$
P intersects with U

Venn diagrams can also represent statements. For example, the Venn diagrams above show the following statements:

X: No police officers wear uniforms.
Y: All police officers wear uniforms.
Z: Some police officers wear uniforms.

Where $P =$ {police officers} and $U =$ {people who wear uniforms}

Key words in statements tell you which type of Venn diagram to draw. See examples below:

Key Words	Type	Examples
"no", "never", "all…do not'	Disjoint sets	No sick pupils come to class. Sick pupils never come to class. All sick pupils do not come to class. (S ={sick pupils} and P ={pupils who come to class} are disjoint)
"all', "no…not", "if…then"	Subset	All football players exercise. There is no football player who does not exercise. If someone is a football player, then they exercise. (F ={football players} is a subset of E ={people who exercise})
"some", "most", "not all"	Intersection	Some pupils use computers. Most pupils use computers. Not all pupils use computers. (P ={pupils} intersects with C ={people who use computers})

You may be asked to determine the validity of given statements. An argument is valid if and only if the conclusion can be drawn from other statements.

Note that the actual **truth does not matter** when determining whether a statement is valid. For example, consider two statements, A: Monrovia is in Sierra Leone, and B: Sierra Leone is in Asia. Although these statements are false, a valid conclusion can be drawn. Based on A and B and using the chain rule, the following is valid: C: Monrovia is in Asia.

Solved Examples

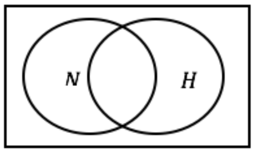
1. For the statements below, $A \Rightarrow B$ and $B \Rightarrow C$. Write as many statements as possible with this information.

   A: Foday injures his leg.
   B: Foday doesn't exercise.
   C: Foday gains weight.

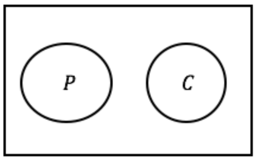
   Solutions:

$A \Rightarrow B$	If Foday injures his leg, then he doesn't exercise.
$B \Rightarrow C$	If Foday doesn't exercise, then he gains weight.
$A \Rightarrow C$	If Foday injures his leg, then he gains weight.
$\sim B \Rightarrow \sim A$	If Foday exercises, then he does not injure his leg.
$\sim C \Rightarrow \sim B$	If Foday does not gain weight, then he does exercise.
$\sim C \Rightarrow \sim A$	If Foday does not gain weight, then he does not injure his leg.

2. Draw a Venn diagram for the statement: A: Some nurses work at the hospital. Let N={nurses} and H={people who work at the hospital}.
   Solution:
   Note that the key word "some" means this is an intersection.

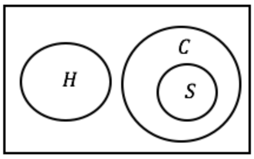


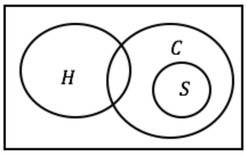
3. Draw a Venn diagram for the statement: No pupils drive cars.
   Solution:
   There are no letters assigned to the sets in the problem, so assign them: P={pupils} and C={people who drive cars}. Note that the key word "no" means these are disjoint sets.



4. The following statements are true for a certain country.
   A: There is **no** civil servant that has a small house.
   B: **All** the civil servants have private cars.

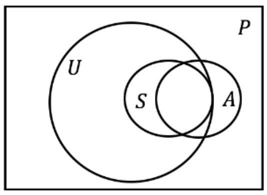
Draw one Venn diagram to show the statements. Let S={Civil servants} $H =$ {People with small houses}, and C= {People with private cars}.
Solution:
"Civil servants" is a subset of "People with private cars". "Civil servants" does not intersect with "people with small houses". They are disjoint. We do not know about the relationship between "People with small houses" and "People with private cars". Therefore, either Venn diagram is correct:





5. Consider the following statements and answer the questions:
   X: All senior secondary pupils wear uniforms.
   Y: Most senior secondary pupils have high attendance.
   a. Draw one Venn diagram to illustrate both statements.
   b. Determine which of the following implications are valid based on X and Y:
      i. Fatu wears a uniform ⇒ Fatu is a senior secondary pupil.
      ii. Michael is a senior secondary pupil ⇒ He has high attendance.
      iii. Hawa does not wear a uniform ⇒ She is not a senior secondary pupil.

   Solutions:
   a. Draw one Venn diagram to illustrate both statements.
      P ={all pupils}, U ={pupils who were uniforms},
      S ={senior secondary pupils}, A ={pupils with high attendance}



b.

i. **Not valid**. All senior secondary pupils wear uniforms. However, all pupils who wear uniforms are not necessarily senior secondary pupils. Fatu may be in U, but outside of S.

ii. **Not valid.** Although most senior secondary pupils have high attendance, not all of them do. Michael might be in set S, but outside of set A.

iii. **Valid.** If Hawa does not wear a uniform, she is outside of set U. She must also be outside of set S, which is a subset of U.

Practice

1. For the statements below, $X \Rightarrow Y$ and $Y \Rightarrow Z$. Write as many statements as possible with this information.

   X: John practises solving Mathematics problems every day.

   Y: John is a very good mathematician.

   Z: John can solve every problem in Mathematics.

2. Consider the following statements, T and S.

   T: People who are teachers.

   S: People who sell goods.

   Prepare a truth table and the related Venn diagram to describe each of the statements below.

   a. A person who is either a teacher or sells goods.
   b. A person who is both a teacher and sells goods.

3. Consider the following statements:

   X: All school inspectors wear uniforms.

   Y: No minister wears a uniform.

   If I = {Inspectors}, U = {People who wear uniforms} and M = {Ministers}, draw a Venn diagram to illustrate the above statements.

4. Consider the following statements:

   X: **All** lazy students are careless.

   Y: **Some** strong students are lazy.

   a. Illustrate the information above on a Venn diagram.
   b. Using the Venn diagram or otherwise, determine whether or not each of the following statements is valid.

      i. Muriel is careless $\Rightarrow$ Muriel is lazy
      ii. Michaela is lazy $\Rightarrow$ Michaela is careless
      iii. Nanday is strong $\Rightarrow$ Nanday is lazy
      iv. Daddy-kay is lazy $\Rightarrow$ Daddy-kay is strong
      v. Ann-Marie is lazy and strong $\Rightarrow$ Ann-Marie is careless
      vi. Angus is careless and strong $\Rightarrow$ Angus is lazy

Lesson Title: Pie charts and bar charts	**Theme:** Probability and Statistics
Practice Activity: PHM4-L038	**Class:** SSS 4

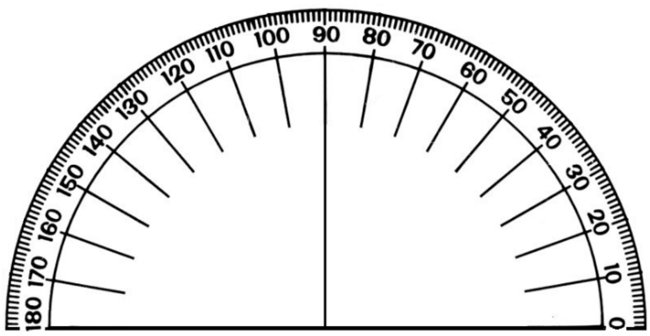
Learning Outcome
By the end of the lesson, you will be able to draw and interpret pie charts and bar charts.

Overview

A pie chart is a type of graph in which a circle is divided into sectors that each represent a portion of the whole. Each sector of the pie chart is a certain percentage of the whole, and the percentages in the chart add up to 100%.

To draw a pie chart accurately, we must use a protractor. The entire circle is 360°, and each segment is one part of the whole. We must find what part of the whole each segment is, and assign a degree to it. Then, we use a protractor to draw an angle inside the pie chart with that degree.

If you do not have a protractor, you can make one with paper. Trace this protractor with a pen onto another piece of paper.



Bar charts are used to compare different quantities. The information displayed in a bar chart corresponds to data in a frequency table. The height of the bars corresponds to the frequency; items with greater frequency have taller bars.

Solved Examples

1. A survey was carried out to find the means of transportation for people in a town. Draw a pie chart for this data.

Transport used	No. of people
Motor bike	16
Bus	20
Taxi	8
Car	12
Boat	4

Solution:

Step 1. Calculate the degree measure for each means of transportation.

The total number of people using the various means of transportation is 60.

Give the number of people for each means of transportation as a fraction of the total number of people using the various forms of transportation (60), and multiply that fraction by 360°.

Motor bike $= \frac{16}{60} \times 360° = 96°$

Bus $= \frac{20}{60} \times 360° = 120°$

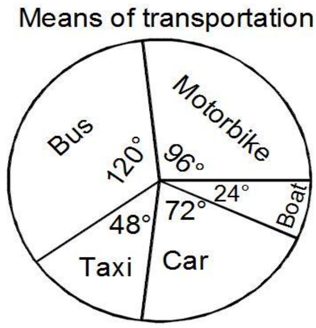
Taxi $= \frac{8}{60} \times 360° = 48°$

Car $= \frac{12}{60} \times 360° = 72°$

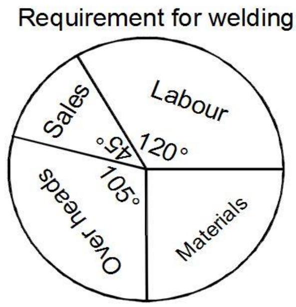
Boat$= \frac{4}{60} \times 360° = 24°$

Step 2. Draw the pie chart using the degrees you found:

- Draw the empty circle with the heading "Means of transportation".
- Draw each segment using a protractor.
- Label each sector with the means of transportation and degrees representing it.



2. A welder records the cost of making iron gates as shown in the pie chart. Use it to answer the questions that follow:
   a. What is the fraction of the total cost represented by i. Overhead and ii. Materials?
   b. What percentage of the total cost is for i. Labour and ii. Sales?
   c. If it costs Le 6,000,000.00 to make a gate, find how much of this is the cost of materials.
   d. If the profit is $6\frac{2}{3}\%$ of the cost of production, find the profit made for producing 5 gates.



Solutions:

a. i. The fraction of the total cost representing overhead is given by the number of degrees representing overhead divided by 360° (sum of the angles in a circle).

Overhead: $\frac{105°}{360°} = \frac{7}{24}$

ii. **Step 1.** Find the angle representing materials:

Angles representing materials= $360° - (105° + 45° + 120°) = 90°$

Step 2. Find materials as a fraction of all costs.

Materials: $\frac{90°}{360°} = \frac{1}{4}$

b. i. Percentage for Labour $= \frac{Degrees\ representing\ Labour}{360°} \times 100$

$$= \frac{120°}{360°} \times 100 = 33\frac{1}{3}\%$$

ii. Percentage for Sales $= \frac{Degrees\ representing\ Sales}{360} \times 100$

$$= \frac{45°}{360°} \times 100 = 12\frac{1}{2}\%$$

c. Cost of materials $= \frac{Degrees\ representing\ Materials}{360°} \times Total\ cost\ of\ production$

$$= \frac{90°}{360°} \times 6{,}000{,}000 = Le\ 1{,}500{,}000.00$$

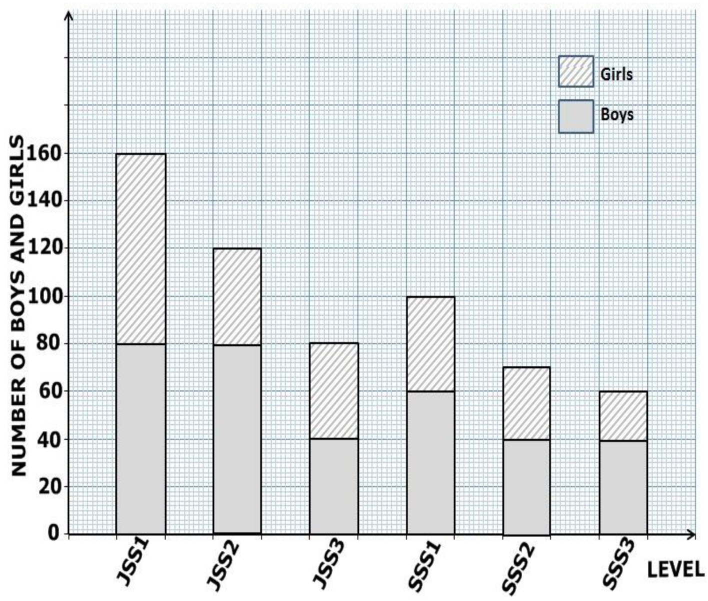
d. Profit for one gate $= 6\frac{2}{3}\% \times cost\ of\ production$

$$= \frac{20}{300} \times Le\ 6{,}000{,}000 = Le\ 400{,}000$$

Profit for five gates$= 5 \times Le\ 400{,}000$

$$= \text{Le } 2{,}000{,}000.00$$

3. The bar chart shows the number of students attending a secondary school in the different levels. Use it to answer the questions that follow:
   a. What is the total number of students in the school?
   b. What is the percentage of girls in the school?
   c. What is the ratio of the number of boys to girls in the JSS level?
   d. What is the probability that a student selected from the school at random is a girl from SSS level?



Solutions:

a. The total number of students is the sum of the heights of the bars:
   Total $= 160 + 120 + 80 + 100 + 70 + 60 = 590$

b. Percentage of girls $= \frac{number\ of\ girls}{Total\ number\ of\ students} \times 100\%$

$$= \frac{80+40+40+40+30+20}{590} \times 100\%$$

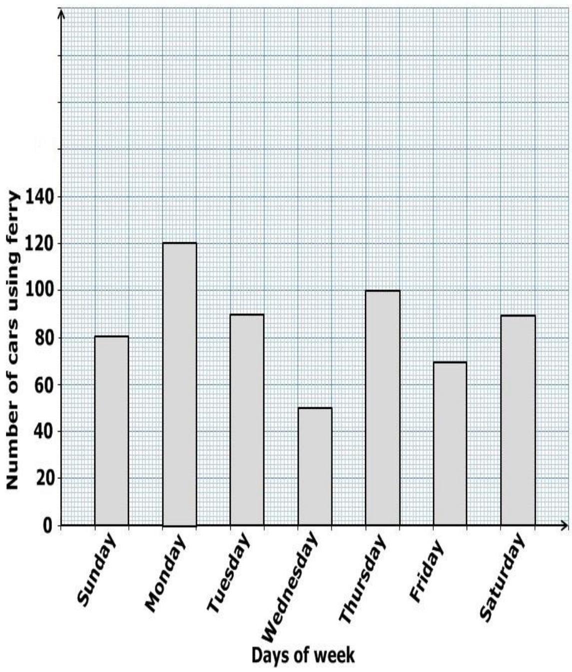
$$= \frac{250}{590} \times 100 = 42.37\%$$

c. Ratio of the number of boys to the number of girls $= \frac{number\ of\ boys\ in JSS}{number\ of\ girls\ in JSS} = \frac{80+80+40}{80+40+40} = \frac{200}{160} = 5:4$

d. Probability $= \frac{number\ girls\ in\ SSS}{Total\ number\ of\ students} = \frac{40+30+20}{590} = \frac{90}{590} = \frac{9}{59}$

Practice

1. The number of cars using a ferry to cross from Kissy terminal to Lungi for each day in a week is shown in the bar chart. Use it to answer the questions below:
    a. What is the total number of cars that cross during the week?
    b. What is the percentage of cars that cross on Wednesday?
    c. How many cars used the ferry on Friday or Saturday?
    d. Which day has the highest number of cars using the ferry?



2. The following are the costs that a company spent on different forms of advertisements in one year:

Requirement	Postal	Computers	Television	Radio	Tape recorders
Degrees	135°	25°	30°	80°	

a. Find the number of degrees representing tape recorders.
b. Draw a pie chart representing the distribution.
c. What is the fraction of the total components which represents: i. Postal, and ii. Tape recorders?
d. What percentage of the total components represent i. Radio, and ii. Computers?
e. If Le 120,000.00 is spent on radio advertisements alone, find i. The total cost for advertising using all components, and ii. The cost for TV advertisements alone.

Lesson Title: Mean, median, and mode of ungrouped data	**Theme:** Probability and Statistics
Practice Activity: PHM4-L039	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to calculate the mean, median, and mode of ungrouped data from lists, tables, and charts.

Overview

The **mean** is a number that can tell us where the middle of the data is. It is also commonly known as the "average". To find the mean of a list of data, add the numbers together and divide the total by the number of items. The quotient is the mean.

For frequency tables or bar charts, mean can be found using the formula $\frac{\sum fx}{\sum f}$, where x represents each value in the data set, and f represents the corresponding frequency of each value. $\sum fx$ can be found by multiplying each frequency by the corresponding value and adding them. $\sum f$ is the total frequency, or the total number of values in the data set.

The number in the middle when the numbers are listed in ascending or descending order is called the **median**. When there is an even number of items in a list, there are 2 numbers in the middle. The median is found by calculating the mean of these 2 numbers.

To find the median in a table or chart, identify the position of the median (for example, in a list of 9 numbers, the 5^{th} number is the median). Identify where the median falls on the table or chart, and give the corresponding number. When the data set has many values, the median position can be found with the formula $\frac{(n+1)}{2}$ where n is the total frequency. For data sets with an even number of values, there are 2 values in the middle, and the median is their average.

The **mode** is the number that appears most often in a list. It can often be easily observed. If no number appears more than once, there is no mode. If multiple numbers appear more than once, there are multiple modes.

The mode is easy to find in a frequency table or bar chart. It is the number with the highest frequency or tallest bar.

Solved Examples

1. A football club plays 30 matches in a season. The number of goals they score for each match is shown below:

$4, 5, 2, 4, 5, 3, 6, 4, 5, 3, 4, 2, 4, 5, 3, 6, 4, 5, 2, 4, 3, 4, 5, 4\ 3, 5,3, 5, 4, 4$

a. Construct a frequency table showing the number of goals scored for each match during the season.
b. Calculate the Mean, Median and Mode.
c. If a score is selected at random, find the probability that it is less than 4.

Solutions:

a. A frequency table doesn't need to have a tally column, but it is helpful to keep count. This frequency table also has a column for fx, which is for frequency multiplied by score.

Score (x)	Tally	Frequency (f)	fx
2	|||	3	6
3	~~||||~~ |	6	18
4	~~||||~~ ~~||||~~ |	11	44
5	~~||||~~ |||	8	40
6	||	2	12
	Total	30	120

b. Use the totals in the frequency table to calculate **mean:** $\frac{\sum fx}{\sum f} = \frac{120}{30} = 4$

Median is the score at the centre after arranging the marks in ascending order. Since the number of marks is even, the median is the mean of the marks at the two middle positions.

We can find the middle position with the formula: $\frac{(n+1)}{2} = \frac{30+1}{2} = 15.5$. In a group of 30, the 15th and 16th scores are in the middle. From the table, these both fall in the row for 4. Therefore, mean is the average of 4 and 4, which is 4.

The **Mode** is the mark with the highest frequency, which is 4.

c. The number of marks below 4 is $6 + 3 = 9$

The probability of a mark less than $4 = \frac{number\ of\ marks\ less\ than\ 4}{total\ number\ of\ marks} = \frac{9}{30} = \frac{3}{10}$

2. The table below shows the marks scored out of 10 by students in a class.

Mark scored (x)	4	5	6	7	8	9	10
No. of students (f)	3	6	8	x	2	1	0

If the mean of the distribution is 6, find the:

a. Value of x
b. Total number of students in the class
c. Median and mode
d. Probability that a student selected at random scored 6 out of 10 marks

Solutions:

a. To solve this problem, we need to substitute all known values in the formula for the mean $\left(\frac{\sum fx}{\sum f}\right)$ and solve for x. Add another row to the table to calculate fx for each value, and add another column to find $\sum f$ and $\sum fx$:

								Total
Mark scored (x)	4	5	6	7	8	9	10	---
No. of students (f)	3	6	8	x	2	1	0	$20+x$
fx	12	30	48	$7x$	16	9	0	$115+7x$

Substitute the values into the formula and solve for x:

$$\begin{aligned}\frac{\sum fx}{\sum f} = \frac{115+7x}{20+x} &= 6\\ 115+7x &= 6(20+x)\\ 115+7x &= 120+6x\\ 7x-6x &= 120-115\\ x &= 5\end{aligned}$$

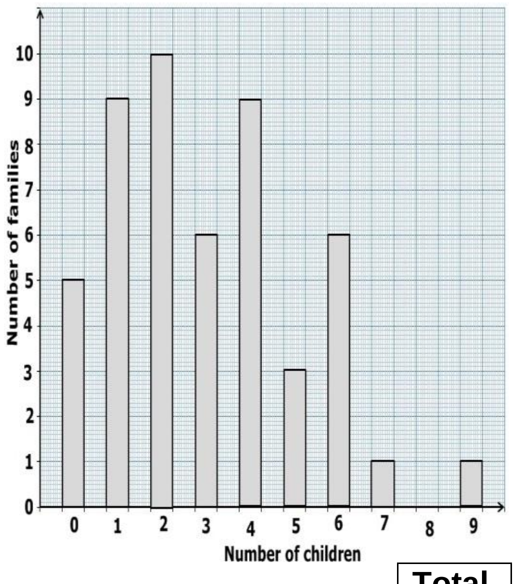
b. The total number of students is the sum of the frequencies:

$$f = 20 + x = 20 + 5 = 25$$

c. **Median** is the mark at the centre when they are arranged in ascending order. In this distribution, the median is the mark at the position $\frac{(n+1)}{2} = \frac{25+1}{2} = 13$. The 13th pupil falls into the column for 6 marks. Therefore, the median is 6. **Mode** is the mark with the highest frequency. From the distribution, the mode is 6.

d. Probability$=\frac{number\ of\ students\ that\ scored\ 6}{total\ number\ of\ students} = \frac{8}{25}$

3. The bar chart shows the number of children per family. Use the chart to answer the questions:
   a. Find the mean, median and mode.
   b. If a family is chosen at random, what is the probability that it is one with 5 children?



Solutions:

a. From the bar chart, we can fill a frequency table as seen below:

Number of children (x)	0	1	2	3	4	5	6	7	8	9	**Total**
Frequency (f)	5	9	10	6	9	3	6	1	0	1	50
fx	0	9	20	18	36	15	36	7	0	9	150

Calculate the **mean:** $\frac{\sum fx}{\sum f} = \frac{150}{50} = 3$

Median: Since the number of families is 50, this gives $\frac{(n+1)}{2} = \frac{50+1}{2} = 25.5$. The median is the average of x at the 25th and 26th positions. This is the average of 3 and 3, which is 3.

Mode is the value of x with the highest frequency. In this case, there are two modes: 1 and 4.

b. **Probability**$= \frac{\textit{number of families with } 5 \textit{ children}}{\textit{total number of families}} = \frac{3}{50}$

Practice

1. The distribution below shows the grades scored (in percentages) by 30 students who took an examination in Mathematics:

   $45, 55, 50, 65, 60, 50, 45, 65, 50, 60, 55, 50, 45, 50, 70,$
   $55, 45, 55, 65, 50, 55, 70, 60, 55, 55, 60, 60, 55, 60, 55$

   a. Construct a frequency table for the distribution.
   b. Calculate the mean of the distribution, correct to 2 decimal places.
   c. Find the median and mode of the distribution.

2. A taxi driver buys fuel for his car every day. The number of litres he bought over a period of some days is given below.

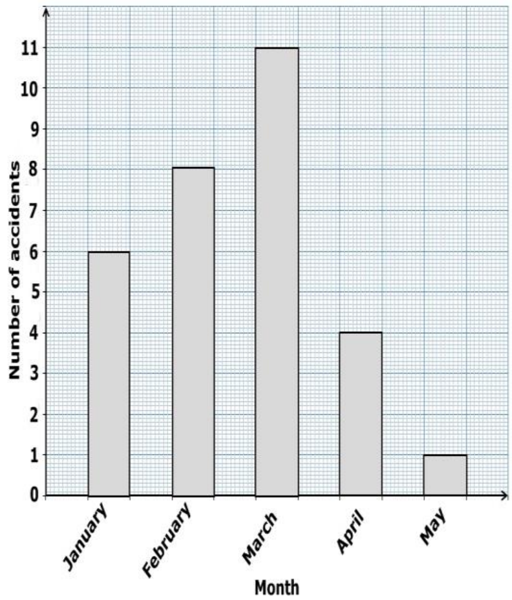
No. of litres	11	12	13	14	15	16
No. of days	4	8	x	12	9	12

   If the mean of the distribution is 14, find the:

   a. Value of x
   b. Number of days he bought fuel
   c. Median and mode

3. The number of fatal accidents between January and May of the year 2015 is shown in the bar chart at right. Use it to answer the questions that follow.

   a. What is the total number of fatal accidents between January and May inclusive?
   b. In which month was there the largest number of fatal accidents?
   c. Calculate the mean number of fatal accidents per month.



Lesson Title: Histograms	**Theme:** Probability and Statistics
Practice Activity: PHM4-L040	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Present grouped data in a frequency distribution table and use it to draw a histogram.
2. Interpret histograms.

Overview

Histograms are similar to bar charts, but they represent grouped data. Grouped data is divided into groups called **class intervals** based on size. A class interval is a group of data with a certain range. Class intervals of a set of grouped data should all have the same range.

Before drawing a histogram, we should first organize grouped data into a frequency table. The first step is to write the list of numbers in ascending order. Then, count the numbers that fall into each class interval. Each class interval is assigned one row of the frequency table.

Like a bar chart, a histogram consists of vertical bars. However, in histograms, the bar does not represent only 1 piece of data, but a range of data. Each bar represents a class interval.

Histograms can be drawn in 2 ways. In the first way, each bar is centred on a **class mid-point** on the x-axis. Class mid-points are the points that lie exactly in the middle of class intervals. In the other way, the high and low values of each class interval are used to graph the intervals. The vertical axis is frequency, which is the same as with bar charts.

The bars of a histogram touch each other, unlike bar charts. Histogram bars touch each other because they represent continuous intervals.

Solved Examples

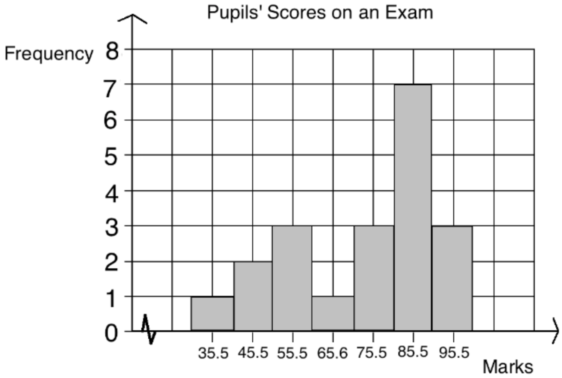
1. The following are marks scored by 20 pupils in an examination: 87, 83, 73, 59, 48, 90, 93, 81, 87, 90, 39, 61, 54, 72, 79, 57, 98, 47, 93, 85.
   a. Draw a frequency table using class intervals 1-10, 11-20, 21-30, …
   b. Draw a histogram for the frequency table.
   c. Which interval do the greatest number of pupils fall into?
   d. If 61% or higher is passing, how many pupils passed?
   e. How many pupils scored 50% or lower?

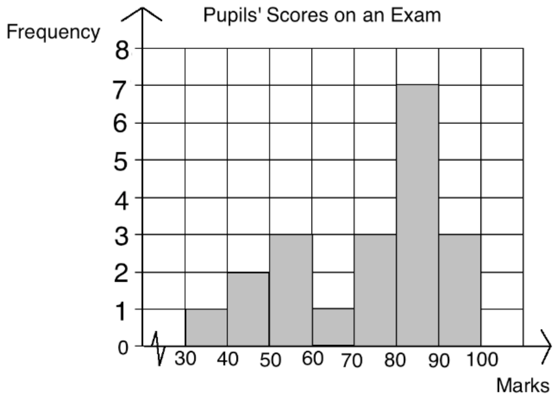
   Solutions:

a. Write the numbers in ascending order before counting and grouping them: 39, 47, 48, 54, 57, 59, 61, 72, 73, 79, 81, 83, 85, 87, 87, 90, 90, 93, 93, 98.

Marks	1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
Frequency	0	0	0	1	2	3	1	3	7	3

b. The histogram can be drawn in 2 ways; bars may be centred at class mid-points, or drawn using class boundaries. Both ways are shown below.
c. The interval 81-90.
d. Add the last 4 frequencies: $1 + 3 + 7 + 3 = 14$ pupils.
e. Add the intervals up to 50: $1 + 2 = 3$ pupils.





The first way is used for the other histograms in this lesson.

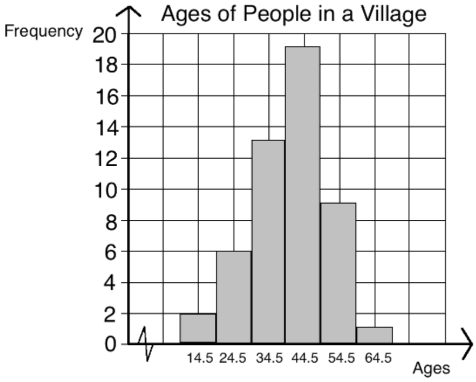
2. The ages in years of people in a village were recorded as follows:
$42, 22, 34, 53, 20, 50, 57, 42, 49, 32, 23, 17, 51, 53, 32, 43, 55, 35, 36, 25,$
$12, 50, 65, 33, 28, 44, 46, 47, 46, 48, 24, 30, 48, 42, 58, 31, 38, 38, 40, 45,$
$34, 36, 39, 40, 42, 42, 45, 46, 48, 59.$
a. Construct a grouped frequency table using class intervals $10 - 19$, $20 - 29$, and so on.
b. How many people are in the village?
c. Draw a histogram for the frequency table.
d. Which age interval has the greatest number of people?
e. What is the percentage of people that are over 29 years old?

Solution:

a.

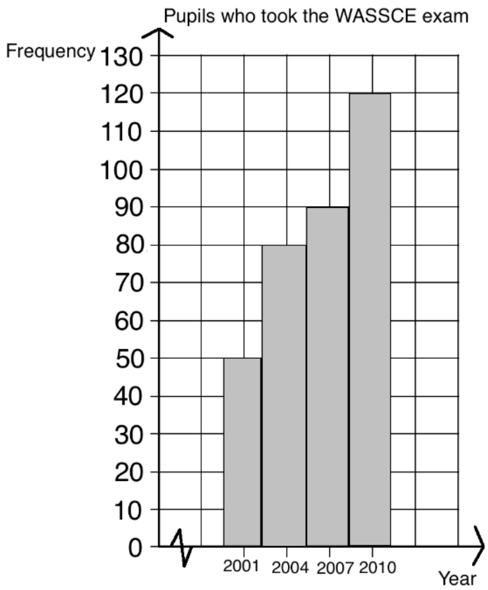
People in a Village	
Ages	**Frequency**
$10 - 19$	2
$20 - 29$	6
$30 - 39$	13
$40 - 49$	19
$50 - 59$	9
$60 - 69$	1

b. Number of people in the village is the sum of the frequencies, which is 50.
c. Histogram:



d. $40 - 49$
e. $\frac{\text{no. of people over 29 years}}{\text{total number of people}} \times 100\% = \frac{13+19+9+1}{50} \times 100\% = 84\%$

3. The histogram shows the number of students who took the WASSCE examination in a school during the years $2000 - 2011$.



a. How many students took the exam during the period $2000 - 2011$ (inclusive of both years)?
b. What is the mode of the distribution?
c. What percentage of the total number of students took the exam during the period $2006 - 2008$ inclusive, correct to 2 places of decimals?
d. What is the probability that a student selected at random took the exam during the period $2003 - 2005$ inclusive?

Solutions:

a. Number of students who took the exams is the sum of the frequencies
$= 50 + 80 + 90 + 120 = 340$
b. Years $2009 - 2011$.
c. $\frac{\text{no.of students who took the exam in } 2006-2008}{\text{total number of students}} \times 100\% = \frac{90}{340} \times 100\% = 26.47\%$
d. No. of students who took the exam during the period $2003 - 2005 = 80$
Probability$= \frac{80}{340} = \frac{4}{17}$

Practice

1. The marks of twenty pupils who took a test in physics are given below:
$15, 1, 6, 10, 13, 3, 12, 7, 13, 5, 12, 7, 11, 9, 8, 14, 10, 18, 17, 21.$

a. Make a frequency table using class intervals of $0-4, 5-9, 10-14$, and so on.
b. Draw a histogram for the frequency table.
c. What is the modal class?
d. If the pass mark is 10, what is the percentage of pupils that passed the test?
e. What is the probability that a pupil selected at random failed the test?

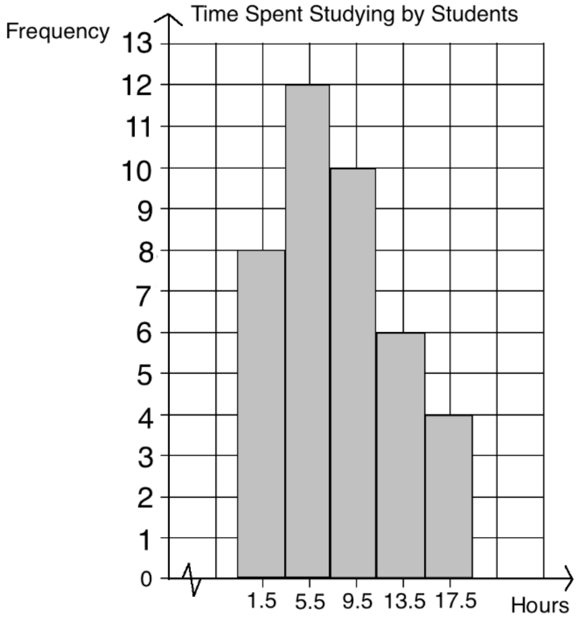
2. The ages of 50 females in a village are recorded as:
$4, 58, 23, 44, 33, 34, 43, 44, 38, 44, 14, 42, 39, 31, 3, 14, 23, 27, 6, 46, 41, 2, 12, 5$
$38, 57, 21, 24, 22, 18, 41, 34, 30, 29, 10, 45, 28, 30, 30, 37, 40, 43, 14, 45, 52, 16$
$30, 2, 19, 34$.
a. Construct a grouped frequency table using class intervals $0-7, 8-15, 16-23$, and so on.
b. Draw a histogram for the frequency table
c. What is the modal class of the distribution?
d. What percentage of females in the village are below 24 years?

3. The height of trees measured in metres by a farmer are given in the frequency table:

Height (m)	$0-6$	$7-13$	$14-20$	$21-27$	$28-34$
No. of trees	6	0	7	3	9

a. Draw a histogram for the distribution
b. What is the total number of trees measured?
c. What is the modal class?
d. What is the percentage of trees below 21 metres?
e. If a tree is selected at random, what is the probability that it is more than 20 m high?

4. The histogram below shows the number of hours a group of students spent studying in one month.



a. How many students are in the group?
b. What is the modal class?
c. How many students studied for less than eight hours?
d. What is the percentage of students who studied for more than eleven hours?

Lesson Title: Frequency polygons	Theme: Probability and Statistics
Practice Activity: PHM4-L041	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to present and interpret grouped data in frequency polygons.

Overview

Recall line graphs, which are used to show ungrouped data. We create line graphs by plotting and connecting points. Frequency polygons are similar to line graphs, in the same way that histograms are similar to bar charts.

Frequency polygons are used to display grouped data, which means that we plot class intervals. We must find each class mid-point on the x-axis, and plot the frequency for the corresponding class interval. Recall that for histograms, we can use either the class mid-points or class boundaries to draw the bars. For frequency polygons, we must use the mid-points.

The class interval that contains the mode is referred to as the "modal class", and the class interval that contains the median is referred to as the "median class". When data is presented as grouped data in a histogram or frequency polygon, we cannot tell exactly where the median or mode is. We can identify the modal class and median class.

For frequency polygons, the highest point gives the modal class. The median class contains the median quantity of the data set. For example, in a set of 7 numbers, the 4^{th} number is the median. We can count up in the frequency polygon to find the interval the 7^{th} number falls into.

Solved Examples

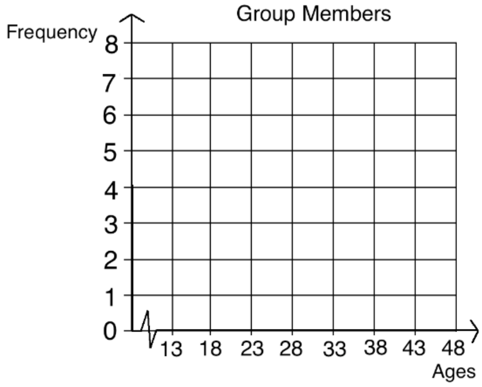
1. A certain women's group has 25 members. Their ages are 21, 42, 35, 26, 32, 19, 23, 27, 29, 38, 41, 42, 27, 35, 18, 30, 31, 26, 24, 41, 22, 35, 37, 23, 20.
   a. Draw a frequency table using class intervals 16-20, 21-25, 26-30, 31-35, 36-40, 41-45.
   b. Draw a frequency polygon.
   c. Identify the modal class and the median class.

   Solutions:
   a. Write the numbers in order: 18, 19, 20, 21, 22, 23, 23, 24, 26, 26, 27, 27, 29, 30, 31, 32, 35, 35, 35, 37, 38, 41, 41, 42, 42.

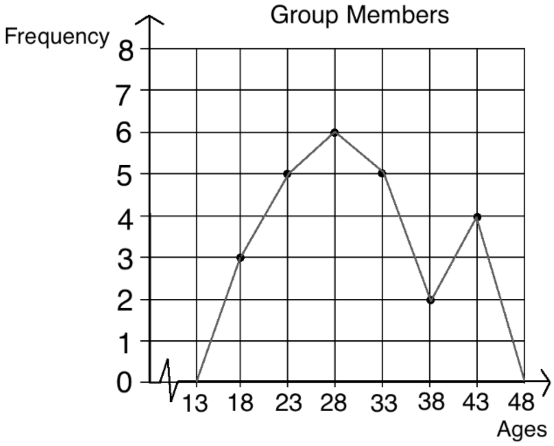
Draw the table:

Group Members	
Ages	**Frequency**
16-20	3
21-25	5
26-30	6
31-35	5
36-40	2
41-45	4

b. Draw the axes for the frequency polygon, using the class mid-points on the x-axis:



Plot the points and connect them. Normally we extend the line of the frequency polygon to the mid-point of what would be the next interval, if that interval existed in the data. In our data set, we don't have any women in the class intervals that contain 13 and 48. Therefore, we can extend the line down to zero.



c. **Modal class:** The highest point gives the modal class. Since the highest point is at 28, the modal class is 26-30. We can also observe this from the frequency table, because it has the greatest frequency.
Median class: The median class contains the age that is in the middle. In a set of 25 women, the 13th woman has the median age. We can count up in the

frequency polygon in the same way that we did for histogram. The 13th woman is in the interval 26-30, so this is the median class.

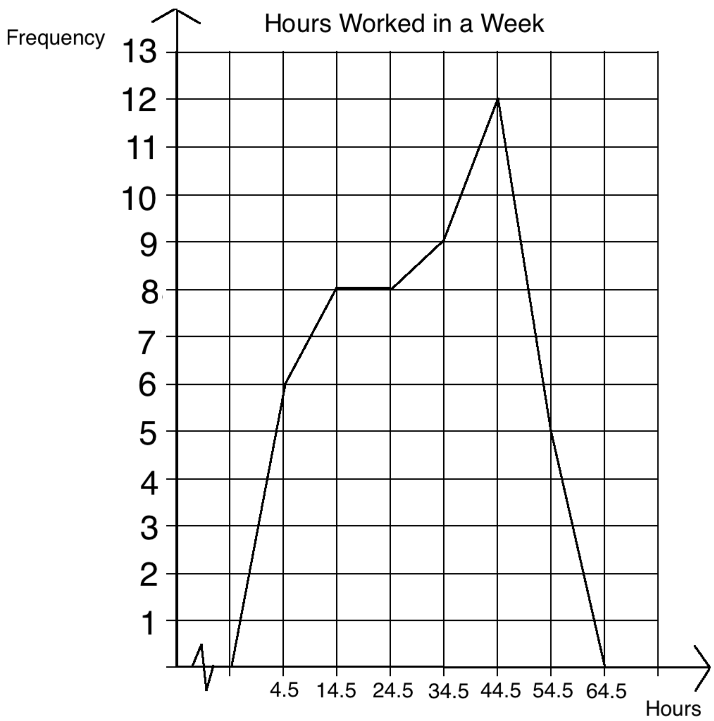
2. The grouped frequency table below shows the number of hours each farmer in a village spent working on his farm in one week.

Farmers in a village	
Hours	**Frequency**
$0-9$	6
$10-19$	8
$20-29$	8
$30-39$	9
$40-49$	12
$50-59$	5

a. Draw a frequency polygon for the distribution.
b. Identify the modal class and the median class.
c. What is the percentage of farmers that worked for less than 20 hours on their farms?

Solutions:

a. Frequency polygon:



b. Modal class: $40-49$, Median class: $30-39$

c. Percentage$=\frac{\text{number that worked for less than 20 hours}}{\text{total number of farmers}}\times 100\%$

$$=\frac{6+8}{50}\times 100\% = 28\%$$

Practice

1. The ages, in years, of 30 students in a class are given as follows:
$18, 8, 12, 20, 14, 16, 13, 16, 9, 17, 11, 15, 13, 9, 17, 19, 12, 17, 8, 18, 12, 20, 16, 21$
$8, 19, 14, 23, 9, 18$.
   a. Make a frequency distribution table, using class intervals $0 - 4, 5 - 9, 10 - 14,$ and so on.
   b. Draw a frequency polygon for the distribution.
   c. State the modal class and the median class.

2. The grades scored by 50 students who took a physics test marked out of 30 are given below:
$18, 6, 5, 13, 0, 25, 26, 20, 9, 8, 13, 2, 8, 4, 26, 7, 27, 3, 28, 22, 23, 24, 23, 8, 21, 11, 26, 13,$
$28, 14, 20, 9, 5, 21, 12, 22, 29, 6, 23, 10, 24, 13, 20, 16, 10, 4, 7, 15, 14, 12$.
   a. Construct a grouped frequency for the distribution using class intervals $0 - 5, 6 - 11, 12 - 17$, and so on.
   b. Draw a frequency polygon for the frequency table.
   c. Give the modal class and the median class.

3. A mechanic has noted the number of vehicles he repaired each day for 45 days in the table below.

No. of vehicles	$5 - 7$	$8 - 10$	$11 - 13$	$14 - 16$	$17 - 19$	$20 - 22$
No. of days	9	10	12	7	5	2

   a. Draw a frequency polygon for the distribution.
   b. State the modal and median classes.
   c. On what percentage of days does he repair 14 or more cars? Give your answer to 1 decimal place.

4. The number of patients treated in a health centre each day is noted in the table below.

No. of patients	$25 - 29$	$30 - 34$	$35 - 39$	$40 - 44$	$45 - 49$	50-54
No. of days	15	25	18	12	15	5

   a. Draw a frequency polygon for the distribution.
   b. What is the modal class?
   c. What is the median class?
   d. On what percentage of days did the centre treat fewer than 30 patients? Give your answer to 2 decimal places.

Lesson Title: Mean, median, and mode of grouped data	Theme: Probability and Statistics
Practice Activity: PHM4-L042	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to calculate the mean, median, and mode of grouped data and apply this to problem solving.

Overview

Calculating the mean, median, and mode of grouped data follows a separate process from ungrouped data. When data is divided into groups, we cannot determine the value of each piece of data in the set. Therefore, we cannot determine the exact mean, median, or mode. We can **estimate** these values. The mean and median are estimated using formulae. The mode is calculated using a histogram.

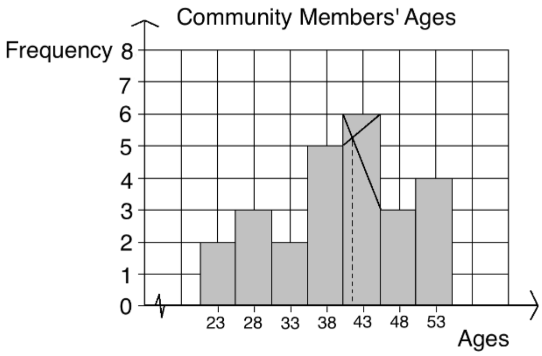
The formula for **estimated mean** is $\bar{x} = \frac{\sum fx}{\sum f}$, where f is frequency, and x is the corresponding class mid-point.

Recall that the sigma symbol (Σ) tells us to find the sum. The numerator tells us to find the sum of each frequency multiplied by each corresponding class mid-point. The denominator tells us to find the sum of the frequencies.

The formula for **estimated median** is $L + \left[\frac{\frac{n}{2} - (\sum f)_L}{f_m}\right] \times c$**,** where L is the lower class boundary of the group containing the median, n is the total frequency of the data, $(\sum f)_L$ is the total frequency for the groups **before** the median group, f_m is the frequency of the median group, and c is the group width.

We can draw a histogram to find the **estimated mode** of grouped data. The class interval that the estimated mode lies in is called the **modal class**. The tallest bar in the histogram is the modal class. We estimate the mode using this bar.

Draw intersecting lines using vertices of the tallest bar, and draw a vertical dotted line to the x-axis. The point where they intersect on the x-axis is the estimated mode. For example, in the histogram, the estimated mode is 41 years old.



Solved Examples

1. In one village, 15 farmers have just harvested their pepper. The table below shows the amount of pepper they harvested in kilograms. Estimate the mean and median using the formulae.

Farmers' Harvests	
Pepper (kg)	**Frequency**
0 – 4	2
5 – 9	5
10 – 14	4
15 – 19	3
20 – 24	1
Total	15

Solution:

Mean: First, find the mid-point of each class interval and add them to the table:

Farmers' Harvests			
Pepper (kg)	**Frequency (f)**	**Mid-point (x)**	fx
0 – 4	2	2	4
5 – 9	5	7	35
10 – 14	4	12	48
15 – 19	3	17	51
20 – 24	1	22	22
Total	15		160

In the numerator, find the sum of the products of each frequency and mid-point. In the denominator, add the frequencies:

$\bar{x} = \frac{\sum fx}{\sum f} = \frac{(2\times2)+(5\times7)+(4\times12)+(3\times17)+(1\times22)}{2+5+4+3+1}$ Substitute the values

$= \frac{4+35+48+51+22}{15}$

$= \frac{160}{15}$

$=$ 10.67 to 2 d.p.

Median $= L + \left[\frac{\frac{n}{2} - (\sum f)_L}{f_m}\right] \times c$ $= 10 + \left[\frac{\frac{15}{2} - (2+5)}{4}\right] \times 5$ Substitute the values

$= 10 + \left[\frac{7.5-7}{4}\right] \times 5$ Simplify

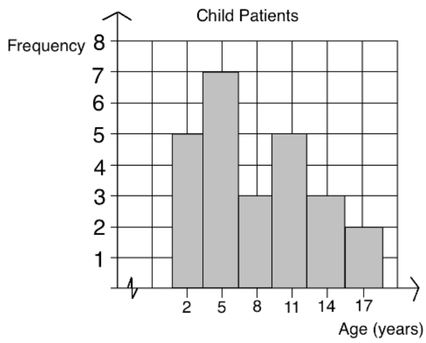
$= 10 + \left[\frac{0.5}{4}\right] \times 5$

$= 10 + 0.125 \times 5$

$= 10 + 0.625$

$= 10.625$

2. The histogram shows the ages of 25 patients in a hospital.
   a. Use the histogram to estimate the mean age of the patients.
   b. Estimate the median age.
   c. Estimate the mode of the patients' ages.



Solutions:

a. Mean:

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{(5\times2)+(7\times5)+(3\times8)+(5\times11)+(3\times14)+(2\times17)}{5+7+3+5+3+2}$$ Substitute the frequencies and mid-points

$$= \frac{10+35+24+55+42+34}{25}$$ Simplify

$$= \frac{200}{25}$$

$= 8$ years old

b. Median:

$$\text{Median} = L + \left[\frac{\frac{n}{2} - (\sum f)_L}{f_m}\right] \times c = 7 + \left[\frac{\frac{25}{2} - (5+7)}{3}\right] \times 3$$ Substitute the values

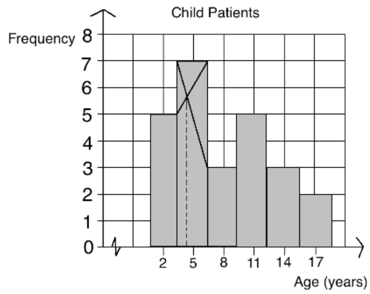
$$= 7 + \left[\frac{12.5-12}{3}\right] \times 3$$ Simplify

$$= 7 + \left[\frac{0.5}{3}\right] \times 3$$

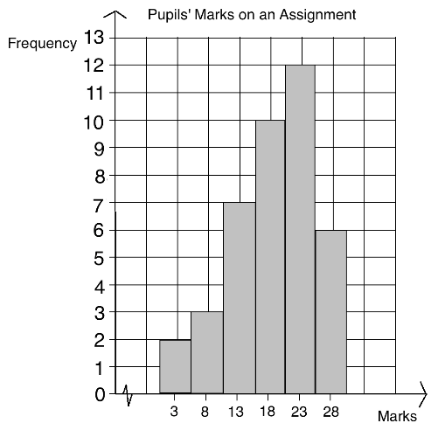
$$= 7 + 0.5$$

$= 7.5$ years old

c. Mode: Draw on the histogram as shown. The estimated mode is 4 years old.

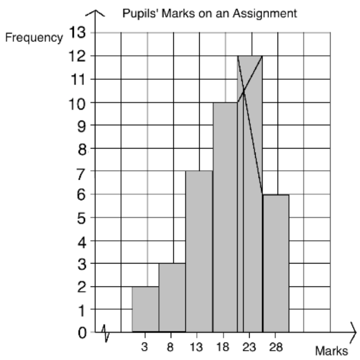


3. The histogram shows the marks 40 pupils received on an assignment.
   a. Estimate the mode of the distribution.
   b. Estimate the mean.
   c. What is the median class?
   d. Estimate the median.



Solutions:

a. Estimate the mode using the histogram. It is approximately 22 marks.



b. Estimated mean:

$$\bar{x} = \frac{\sum fx}{\sum f} = \frac{(2\times3)+(3\times8)+(7\times13)+(10\times18)+(12\times23)+(6\times28)}{2+3+7+10+12+6}$$ Substitute the frequencies and mid-points

$$= \frac{6+24+91+180+276+168}{40}$$ Simplify

$$= \frac{745}{40}$$

$$= 18.63$$

c. The median is the mean of the 20th and 21st scores. Pupils 20 and 21 fall into the interval 16-20, which is the median class.

d. Estimated median:

$$\text{Median} = L + \left[\frac{\frac{n}{2} - (\sum f)_L}{f_m}\right] \times c = 16 + \left[\frac{\frac{40}{2} - (2+3+7)}{10}\right] \times 5$$ Substitute the values

$$= 16 + \left[\frac{20-12}{10}\right] \times 5$$ Simplify

$$= 16 + \left[\frac{8}{10}\right] \times 5$$

$$= 16 + 4$$

$$= 20$$

Practice

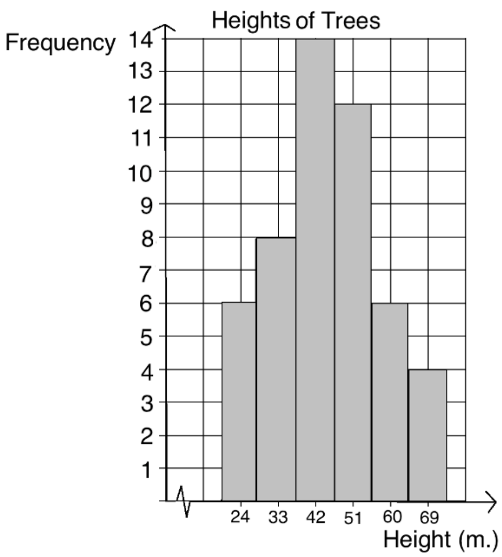
1. The marks scored by 20 students in an examination are given in the frequency table below. Estimate, correct to 1 decimal place, the: a. mean; b. median.

Marks	$0-4$	$5-9$	$10-14$	$15-19$	$20-24$	$25-29$	$30-34$
No. of students	2	5	2	7	2	1	1

2. The distribution below gives the weights (in kg) of bags in a store. Estimate the mean and median of the distribution.

Weight (kg)	$10-21$	$22-33$	$34-45$	$46-57$	$58-69$
No. of bags	15	5	21	10	29

3. The histogram below shows the heights (in metres) of trees in a garden. Estimate, correct to two places of decimal the: a. mean; b. median; c. mode.



Lesson Title: Cumulative frequency curves and quartiles	**Theme:** Probability and Statistics
Practice Activity: PHM4-L043	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Construct a cumulative frequency curve and estimate quartiles.
2. Calculate inter-quartile range and semi inter-quartile range.

Overview

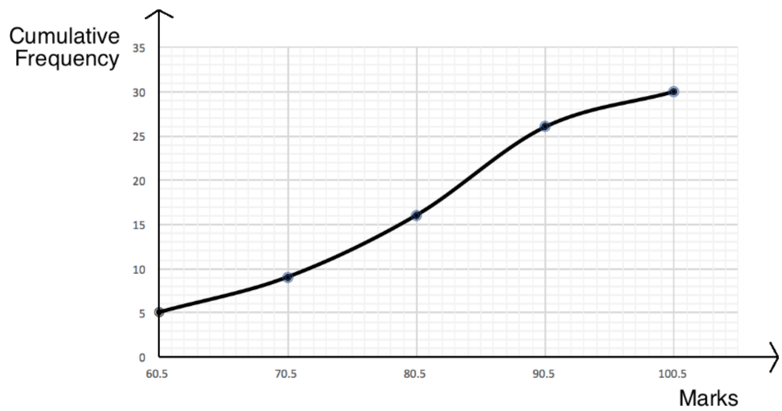
This lesson is on cumulative frequency curves, which are constructed based on cumulative frequency tables. A cumulative frequency table is one with class intervals, where the cumulative frequency for each row is calculated by adding that row's frequency to the cumulative frequency for rows above it. See the example below.

A **cumulative frequency (c.f.) curve** can be graphed in a similar way to a line graph. Cumulative frequency curves can also be called **"ogive"**.

A c.f. curve increases as it moves in the positive direction along the x-axis. For the x-values, we will plot the upper class boundary of each class interval. This is the highest data point in each class interval. In the table below, notice that there is a space of 1 unit between each interval. The first class interval ends at 60, and the second class interval begins at 61. For the purpose of graphing, we will take the point in the middle of the class intervals. For example, we will plot the value 60.5. For the y-value, we will plot the cumulative frequency from the table.

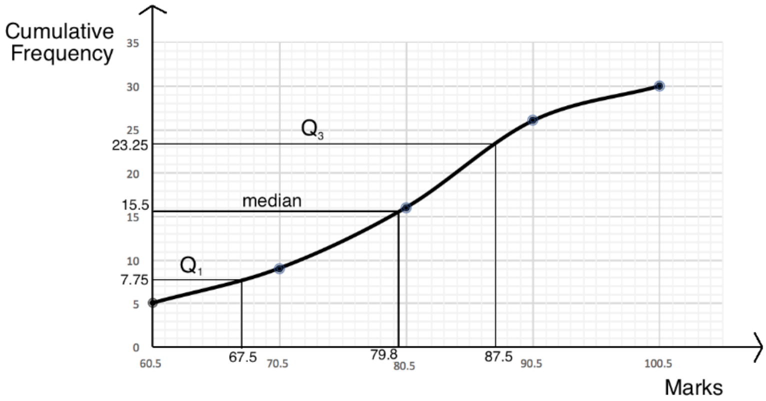
Pupils' Scores on a Maths Test			
Marks	**Frequency**	**Cumulative Frequency**	**Upper Class Boundary**
51 – 60	5	5	60.5
61 – 70	4	5 + 4 = 9	70.5
71 – 80	7	7 + 9 = 16	80.5
81 – 90	10	10 + 16 = 26	90.5
91 – 100	4	4 + 26 = 30	100.5
Total	30		

Plot the points and connect them with a smooth curve:



Quartiles divide a data set into 4 equal parts. The lower quartile (Q_1) is one-quarter of the way from the bottom of the data. The upper quartile (Q_3) is one-quarter of the way from the top of the data. The second quartile (Q_2) is the median, or the middle quartile. When we find quartiles from grouped data, the results are only estimates.

Estimate the quartiles by first finding their placement in the dataset, then using the c.f. curve. The quartiles are located at $Q_1: \frac{1}{4}(n+1)$, $Q_2: \frac{1}{2}(n+1)$ and $Q_3: \frac{3}{4}(n+1)$, where n is the total frequency. The quartiles of the example c.f. curve are shown below. Their estimated values are $Q_1 = 67.5$ marks, $Q_2 = 79.8$ marks, $Q_3 = 87.5$ marks.



Just as we can calculate the range of a data set, we can calculate the interquartile range. The interquartile range represents how spread out the middle half of the data is. It is found by subtracting the lower quartile from the upper quartile ($Q_3 - Q_1$). For this dataset, the interquartile range is $87.5 - 67.5 = 20$ marks.

The semi-interquartile range tells us about one quarter of the data set ("semi" means half, so it is half of the interquartile range). The semi-interquartile range is given by the formula: $Q = \frac{Q_3 - Q_1}{2}$. For this data set, the semi-interquartile range is $Q = \frac{87.5 - 67.5}{2} = \frac{20}{2} = 10$ marks. This tells us that about half of the pupils scored within 10 marks of the median score.

Solved Examples

1. The table below gives the marks of 25 pupils on a Maths test.
    a. Fill the empty columns.
    b. Draw the cumulative frequency curve.
    c. Use the curve to estimate the median mark.
    d. Use the curve to estimate the upper and lower quartiles.
    e. Calculate the interquartile range.
    f. Calculate the semi-interquartile range.

Pupils' Scores on a Maths Test			
Marks	**Frequency**	**Upper Class Boundary**	**Cumulative Frequency**
51 – 60	3		
61 – 70	5		
71 – 80	7		
81 – 90	6		
91 – 100	4		
Total	25		

Solutions:

a. Completed table:

Pupils' Scores on a Maths Test			
Marks	**Frequency**	**Upper Class Boundary**	**Cumulative Frequency**
51 – 60	3	60.5	3
61 – 70	5	70.5	5 + 3 = 8
71 – 80	7	80.5	7 + 8 = 15
81 – 90	6	90.5	15 + 6 = 21
91 – 100	4	100.5	21 + 4 = 25
Total	25		

b. See below.

c. Median position: $\frac{1}{2}(n+1) = \frac{1}{2}(25+1) = \frac{1}{2}(26) = 13$; the 13th pupil has the median score.

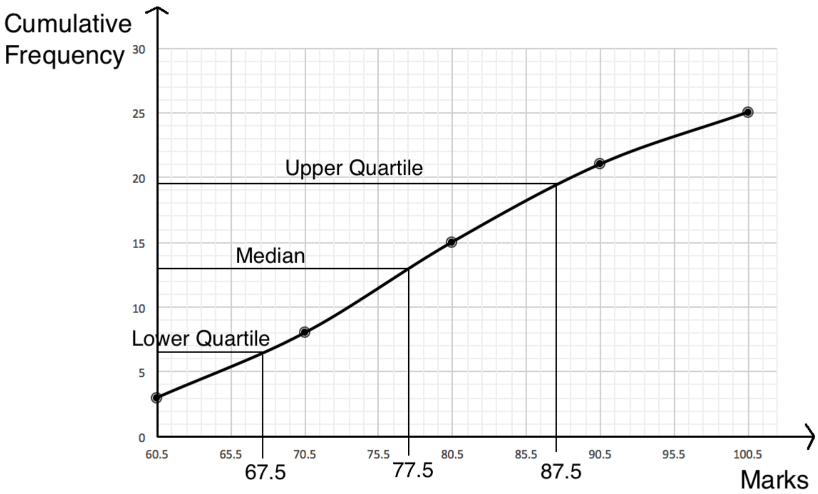
Estimated median = 77.5 (see curve below)

d. Q_1 position: $\frac{1}{4}(n+1) = \frac{1}{4}(25+1) = \frac{1}{4}(26) = \frac{26}{4} = 6\frac{1}{2}$

Estimated $Q_1 = 67.5$

Q_3 position: $\frac{3}{4}(n+1) = \frac{3}{4}(25+1) = \frac{3}{4}(26) = \frac{78}{4} = 19\frac{1}{2}$

Estimated $Q_3 = 87.5$



e. Interquartile range: $Q_3 - Q_1 = 87.5 - 67.5 = 20$ marks

f. $Q = \frac{Q_3 - Q_1}{2} = \frac{87.5-67.5}{2} = \frac{20}{2} = 10$ marks

2. The cassava harvests of 25 farmers are shown in the frequency table:

Harvest (kg)	31-40	41-50	51-60	61-70	71-80
Frequency	4	8	6	4	3

a. Construct a cumulative frequency table for the distribution.

b. Use the table to draw a cumulative frequency curve.

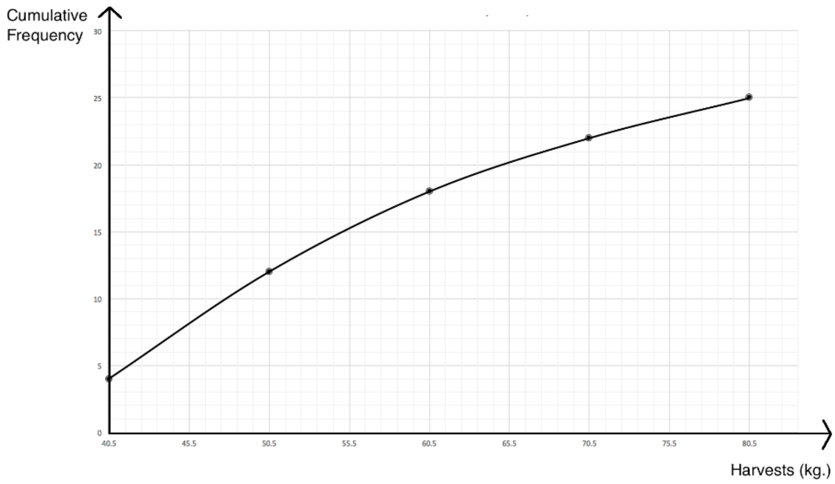
Solutions:

a. Cumulative frequency table:

Cassava harvests			
Marks	**Frequency**	**Upper Class Boundary**	**Cumulative Frequency**

31-40	4	40.5	4
41-50	8	50.5	4+8=12
51-60	6	60.5	12+6=18
61-70	4	70.5	18+4=22
71-80	3	80.5	22+3=25
Total	25		

b. Cumulative frequency curve:



Practice

1. The table below gives the distribution of the ages (in years) of people in a village who are over 44 years old.

Age	$45-49$	$50-54$	$55-59$	$60-64$	$65-69$	$70-74$
Frequency	60	35	25	15	10	5

   a. Construct a cumulative frequency table for the distribution.
   b. Draw a cumulative frequency curve for the distribution

2. The table shows the frequency distribution of the marks scored by 60 students in an examination.

Mark (%)	$40-45$	$46-51$	$52-57$	$58-63$	$64-69$	$70-75$
Frequency	4	6	20	15	10	5

   a. Prepare the cumulative frequency table and use it to draw the cumulative frequency curve.
   b. Use the curve to estimate the: i. lower and upper quartiles; ii. median mark; iii. inter-quartile range.

3. The thickness of each piece of metal in a workshop is measured and recorded.

Thickness (mm)	$1-3$	$4-6$	$7-9$	$10-12$	$13-15$	$16-18$
No. of pieces	8	16	30	15	17	4

   a. Make a cumulative frequency table and use it to draw the cumulative frequency curve.
   b. Use your graph to estimate the: i. lower and upper quartiles; ii. median thickness; iii. semi-interquartile range.

Lesson Title: Percentiles	Theme: Probability and Statistics
Practice Activity: PHM4-L044	**Class:** SSS 4

Learning Outcomes
By the end of the lesson, you will be able to:
1. Estimate percentiles of data from the cumulative frequency curve.
2. Apply percentiles to real-life problems.

Overview

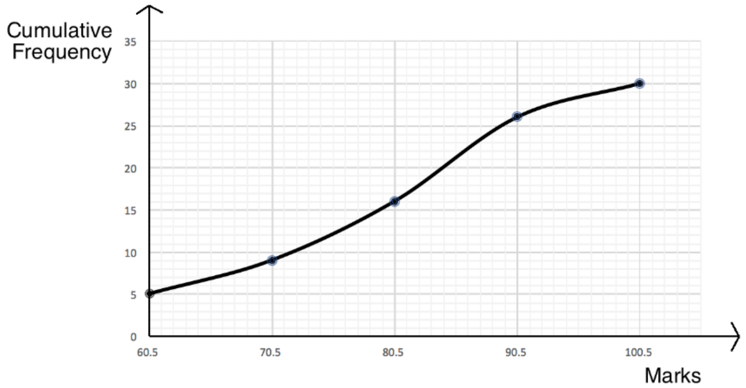
Percentiles are estimated using the cumulative frequency curve, in a similar way to quartiles. Percentiles divide the data set into 100 equal parts. For example, the 30th percentile divides off the lowest 30% of the data.

We use a formula to find the position of a quartile, then use the cumulative frequency curve to identify its value.

The nth percentile is the mark at $\frac{n}{100}\sum f$. This is the formula that gives a percentile's position. After using the formula, find the position on the y-axis of the curve. Draw horizontal and vertical lines to estimate the corresponding value on the x-axis. This is the estimated percentile.

Solved Examples

1. This cumulative frequency curve shows the marks that 30 pupils received on an exam. Find the value of the 30th percentile.

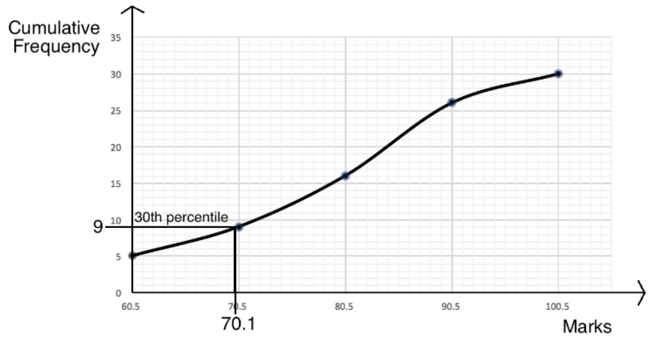


Solution:

Use the formula to find the position of the 30th percentile:

$$\frac{n}{100}\sum f = \frac{30}{100}(30) = \frac{900}{100} = 9$$

Estimate the 30th percentile using the curve, as shown. The 30th percentile is 70.1 marks.



2. The table below gives the weights of 100 individuals.

Weights (kg)	25-29	30-34	35-39	40-44	45-49	50-54	55-59	60-64
Frequency	1	9	10	22	27	14	11	6

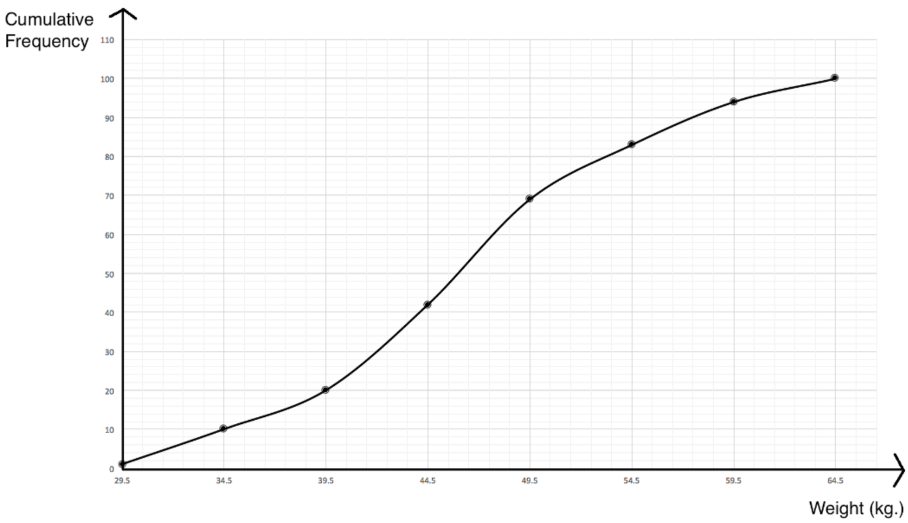
a. Construct a cumulative frequency table.
b. Use the table to draw the cumulative frequency curve.
c. Use the curve to estimate the following: i. 15th percentile; ii. 60th percentile; iii. 95th percentile

Solutions:

a. Cumulative frequency table:

Weights			
Weight (kg)	**Frequency**	**Upper Class Boundary**	**Cumulative Frequency**
25 – 29	1	29.5	1
30 – 34	9	34.5	1+9=10
35 – 39	10	39.5	10+10=20
40 – 44	22	44.5	20+22=42
45 – 49	27	49.5	42+27=69
50 – 54	14	54.5	69+14=83
55 – 59	11	59.5	83+11=94
60 – 64	6	64.5	94+6=100
Total	100		

b. Cumulative frequency curve:

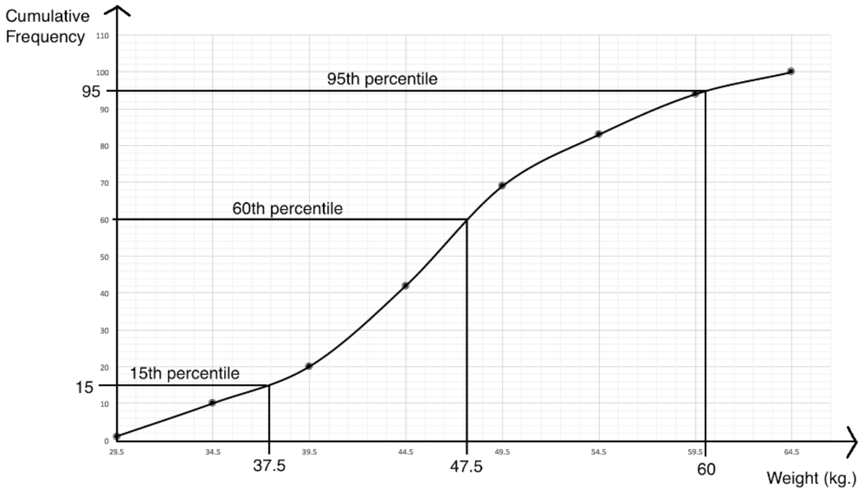


c. Estimations of percentiles (see graph below):

i. Position: $\frac{n}{100}\sum f = \frac{15}{100}(100) = \frac{1500}{100} = 15$; 15th percentile $= 37.5$ kg

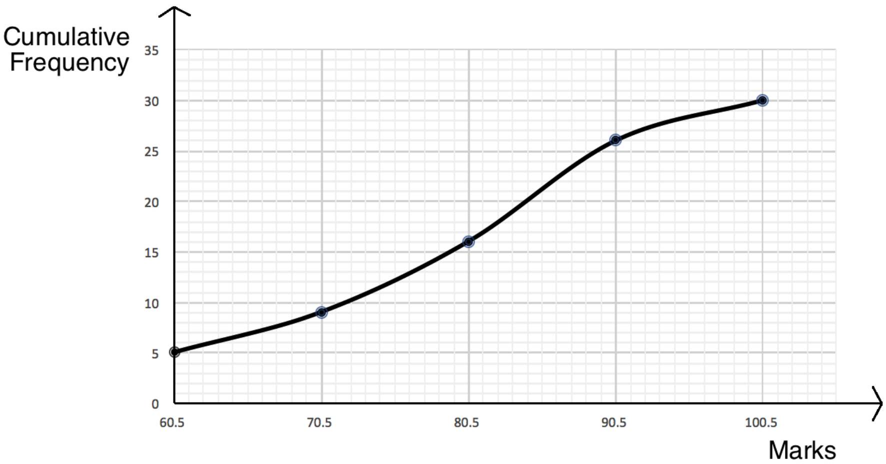
ii. Position: $\frac{n}{100}\sum f = \frac{60}{100}(100) = 60$; 60th percentile $= 47.5$ kg

iii. Position: $\frac{n}{100}\sum f = \frac{95}{100}(100) = 95$; 95th percentile $= 60$ kg



Practice

1. The cumulative frequency curve below gives the distribution of marks that 30 pupils scored on a Maths exam. Use the curve to estimate the following percentiles: a. 35th; b. 70th; c. 90th.



2. The scores of 50 pupils on a test are shown in the frequency table below.

Marks (%)	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
Frequency	3	4	3	6	7	10	12	5

Draw a cumulative frequency curve for the distribution, and use it to estimate the: a. 30th percentile; b. 45th percentile; c. 90th percentile.

3. The table below gives the age distribution of all pupils enrolled in primary and secondary school in a village.

Age (years)	5-8	9-11	12-14	15-17	18-20
Frequency	20	32	18	14	6

a. Construct a cumulative frequency table.
b. Use the table to draw the cumulative frequency curve.
c. Use the curve to estimate the 80^{th} percentile.
d. If the youngest 40% of the pupils are eligible for an early education programme, estimate how many pupils are eligible.
e. Estimate the age of the oldest pupil eligible for the programme.
f. If the oldest 30% of pupils are eligible for a technical training programme, estimate how many pupils are eligible.
g. Estimate the age of the youngest pupil eligible for the technical training programme.

4. Bentu manages a chimpanzee sanctuary for rescued chimpanzees. The table below gives the weights of the 50 chimpanzees living at the sanctuary.

Weight (kg)	10-19	20-29	30-39	40-49	50-59	60-69
Frequency	4	9	12	15	8	2

a. Construct a cumulative frequency table.
b. Use the table to draw the cumulative frequency curve.
c. Bentu designed one living space for the smallest 30% of chimpanzees. Estimate how many chimpanzees live in that space.
d. The heaviest 40% of chimpanzees live in a very large space outdoors. Estimate how many chimpanzees live in that space.
e. Estimate the weight of the smallest chimpanzee living in the large space outdoors.

Lesson Title: Measures of dispersion	**Theme:** Probability and Statistics
Practice Activity: PHM4-L045	**Class:** SSS 4

Learning Outcomes

By the end of the lesson, you will be able to:

1. Describe and interpret the dispersion or spread of values in a data set.
2. Calculate the range and variance of a set of ungrouped values.

Overview

This lesson is on dispersion, which is related to how spread out the data is. There are many different ways to measure the dispersion of a data set, but this lesson focuses on range, deviation, and variance.

Range is one of the most common ways to measure dispersion. Recall that range is the difference between the greatest and least values in a data set, which is found using subtraction. The larger the range, the more spread out the data set is. This is true for all measures of dispersion. If a measure of dispersion has greater value, the data is more spread out.

Deviation is another measure of dispersion, which is applied to an individual piece of data in the set. The deviation of a value gives its distance from the mean. To calculate deviation, subtract mean from the given data point.

Variance is another measure of dispersion. Variance is calculated using the deviations of the data. The deviations of the data tell you information about each piece of data, and this information can be used to obtain a single number which indicates the overall dispersion of the data. To calculate variance, find the sum of the square of each deviation from the mean. Then, divide by the frequency. See Solved Example 2 for a calculation of variance.

Solved Examples

1. The ages of 20 university students are 18, 18, 18, 19, 19, 19, 19, 20, 20, 20, 20, 21, 21, 21, 22, 22, 23, 23, 24, and 25.
   a. Calculate the range of the data.
   b. Calculate the mean.
   c. Calculate the deviation of a 24-year-old student.
   d. Calculate the deviation of an 18-year-old student.

   Solutions:

   a. Range = greatest value – least value $= 25 - 18 = 7$ years
   b. Mean $= \frac{\text{sum of ages}}{\text{number of students}} = \frac{3(18)+4(19)+4(20)+3(21)+2(22)+2(23)+24+25}{20} = \frac{412}{20} = 20.6$ years old
   c. Deviation $= 24 - \text{mean} = 24 - 20.6 = +3.4$

d. Deviation = $18 - \text{mean} = 18 - 20.6 = -2.6$

2. Six children living in one household have weights (in kilogrammes) of 20, 24, 51, 30, 16 and 21 kg. Calculate:
   a. Range
   b. Mean
   c. Variance

   Solutions:
   a. Range = $51 - 16 = 35$ kg
   b. Mean = $\frac{\text{sum of weights}}{\text{number of children}} = \frac{20+24+51+30+16+21}{6} = \frac{162}{6} = 27$ kg
   c. Variance:
      Step 1. Find the deviation of each value in the data set:
      - $20 - \text{mean} = 20 - 27 = -7$
      - $24 - \text{mean} = 24 - 27 = -3$
      - $51 - \text{mean} = 51 - 27 = +24$
      - $30 - \text{mean} = 30 - 27 = +3$
      - $16 - \text{mean} = 16 - 27 = -11$
      - $21 - \text{mean} = 21 - 27 = -6$

      Step 2. Calculate variance:

$$\text{Variance} = \frac{(-7)^2+(-3)^2+(24)^2+(3)^2+(-11)^2+(-6)^2}{6}$$
$$= \frac{49+9+576+9+121+36}{6}$$
$$= \frac{800}{6}$$
$$= 133.3$$

3. Ten players on a football team need new shoes. Their shoe sizes are 41, 39, 44, 41, 40, 44, 45, 40, 42, and 44. Calculate:
   a. The range in shoe sizes.
   b. The mean shoe size.
   c. The variance in shoe sizes.

   Solutions:
   a. Range = $45 - 39 = 6$ sizes
   b. Mean = $\frac{\text{sum of sizes}}{\text{number of players}} = \frac{41+39+44+41+40+44+45+40+42+44}{10} = \frac{420}{10} = 42$
   c. Variance:
      Step 1. Find the deviation of each value in the data set:
      - $41 - \text{mean} = 41 - 42 = -1$
      - $39 - \text{mean} = 39 - 42 = -3$
      - $44 - \text{mean} = 44 - 42 = +2$
      - $41 - \text{mean} = 41 - 42 = -1$
      - $40 - \text{mean} = 40 - 42 = -2$
      - $44 - \text{mean} = 44 - 42 = +2$

- $45 - \text{mean} = 45 - 42 = +3$
- $40 - \text{mean} = 40 - 42 = -2$
- $42 - \text{mean} = 42 - 42 = 0$
- $44 - \text{mean} = 44 - 42 = +2$

Step 2. Calculate variance:

$$\text{Variance} = \frac{(-1)^2+(-3)^2+(+2)^2+(-1)^2+(-2)^2+(+2)^2+(+3)^2+(-2)^2+(0)^2+(+2)^2}{10}$$
$$= \frac{1+9+4+1+4+4+9+4+0+4}{10}$$
$$= \frac{40}{10}$$
$$= 10$$

Practice

1. The heights of 6 pupils in centimetres are 120, 122, 124, 127, 127, and 130. Calculate:
    a. The range in height.
    b. The mean height.
    c. The variance of their heights.
2. The scores that 10 pupils achieved on an exam are 68, 83, 70, 72, 84, 69, 88, 75, 78, and 83. Calculate the range and variance of their scores.
3. The weights of 8 goats in kilogrammes are 20, 24, 30, 22, 27, 31, 27, and 35. Calculate:
    a. The range in weight.
    b. The mean weight.
    c. The variance of their weights.

Lesson Title: Standard deviation	**Theme:** Probability and Statistics
Practice Activity: PHM4-L046	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to calculate the standard deviation of ungrouped and grouped data.

Overview

Standard deviation is a measure of dispersion that tells us how close data points generally are to the mean. A low standard deviation indicates that points are generally close to the mean (the deviation is low). A high standard deviation indicates that points are spread out over a wide range of values, farther from the mean (the deviation is high).

Ungrouped data: $s = \sqrt{\frac{1}{n}\sum(x-\bar{x})^2}$, where n is the sum of the frequencies, x is each value in the set, and $\bar{x}$ is the mean.

Grouped data: $s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2}$, where f is frequency, x is each data point, and $\bar{x}$ is the mean.

We need to take several steps to solve each problem. We will use a table to organise these calculations. When drawing a table, it is important to have a column for each term that is needed in the formula.

Notice that each formula requires us to find the mean. The mean does not get a column in the table. We will calculate the mean of each data set separately.
For grouped data with class intervals, find the mid-point of each interval. The mid-points are used for x in the standard deviation formula.

Solved Examples

1. 10 pupils achieved the following scores on a Maths exam: 80, 82, 88, 89, 84, 79, 81, 82, 85, 80. Calculate the standard deviation of the distribution.
   Solution:
   Step 1. Calculate the mean: $\bar{x} = \frac{80+82+88+89+84+79+81+82+85+80}{10} = \frac{830}{10} = 83$
   Step 2. Fill the table (below) to find $\sum(x-\bar{x})^2$.
   Step 3. Apply the formula: $s = \sqrt{\frac{1}{n}\sum(x-\bar{x})^2} = \sqrt{\frac{106}{10}} \approx 3.26$

x	$x-\bar{x}$	$(x-\bar{x})^2$
80	$80-83=-3$	9

82	$82 - 83 = -1$	1
88	$88 - 83 = +5$	25
89	$89 - 83 = +6$	36
84	$84 - 83 = +1$	1
79	$79 - 83 = -4$	16
81	$81 - 83 = -2$	4
82	$82 - 83 = -1$	1
85	$85 - 83 = +2$	4
80	$80 - 83 = -3$	9
Total	$= \sum(x - \bar{x})^2 =$	106

2. The ages of 20 children are given in the table below. Calculate the mean and standard deviation of their ages.

Age (years)	1	2	3	4	5	6
Frequency (f)	3	4	2	3	6	2

Solution:

Step 1. Draw and fill a table with the values needed for the mean and standard deviation formulae. Notice that the last column, fx^2, is the product of the first and third columns: $xfx = fx^2$.

$\boldsymbol{x}$	$\boldsymbol{f}$	$\boldsymbol{fx}$	$\boldsymbol{fx^2}$
1	3	$1 \times 3 = 3$	$1 \times 3 = 3$
2	4	$2 \times 4 = 8$	$2 \times 8 = 16$
3	2	$3 \times 2 = 6$	$3 \times 6 = 18$
4	3	$4 \times 3 = 12$	$4 \times 12 = 48$
5	6	$5 \times 6 = 30$	$5 \times 30 = 150$
6	2	$6 \times 2 = 12$	$6 \times 12 = 72$
Totals	$\sum f = 20$	$\sum fx = 71$	$\sum fx^2 = 307$

Step 2. Calculate mean: $\bar{x} = \frac{\sum fx}{\sum f} = \frac{71}{20} = 3.55$ years old

Step 3. Calculate standard deviation: $s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2} = \sqrt{\frac{307}{20} - (3.55)^2} =$
$\sqrt{15.35 - 12.60} = \sqrt{2.75} \approx 1.66$

3. The table below shows the amount of cassava sold in one week by 25 saleswomen. Use the table to calculate the mean and standard deviation.

Sales (kg)	1	2	3	4	5	6
Frequency (f)	3	5	4	6	3	4

Solution:

Complete a solution table with each term needed for the mean and standard deviation formulae:

x	f	fx	fx^2
1	3	$1 \times 3 = 3$	$1 \times 3 = 3$
2	5	$2 \times 5 = 10$	$2 \times 10 = 20$
3	4	$3 \times 4 = 12$	$3 \times 12 = 36$
4	6	$4 \times 6 = 24$	$4 \times 24 = 96$
5	3	$5 \times 3 = 15$	$5 \times 15 = 75$
6	4	$6 \times 4 = 24$	$6 \times 24 = 144$
Totals	$\sum f = 25$	$\sum fx = 88$	$\sum fx^2 = 374$

Calculate mean: $\bar{x} = \frac{\sum fx}{\sum f} = \frac{88}{25} = 3.52$ kg

Calculate the standard deviation: $s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2} = \sqrt{\frac{374}{25} - (3.52)^2} =$ $\sqrt{14.96 - 12.39} = \sqrt{2.57} \approx 1.60$

4. The table below shows the distribution of marks scored by students in an examination. Calculate, correct to 2 decimal places, the:
   a. mean
   b. standard deviation

Marks	60-64	65-69	70-74	75-79	80-84	85-89
Frequency (f)	1	2	6	7	3	1

Solution:

For grouped data with class intervals, we must find the mid-point of each interval. The mid-points will be used for the value of variable x.

Complete the table with each term needed for the mean and standard deviation formulae:

Interval	Mid-point (x)	f	fx	fx^2
60-64	62	1	62	3,844
65-69	67	2	134	8,978
70-74	72	6	432	31,104
75-79	77	7	539	41,503
80-84	82	3	246	20,172
85-89	87	1	87	7,569
	Totals	$\sum f = 20$	$\sum fx = 1500$	$\sum fx^2 = 113{,}170$

Calculate mean: $\bar{x} = \frac{\sum fx}{\sum f} = \frac{1500}{20} = 75$ marks

Calculate standard deviation: $s = \sqrt{\frac{\sum fx^2}{\sum f} - \bar{x}^2} = \sqrt{\frac{113{,}170}{20} - 75^2} =$ $\sqrt{5{,}658.5 - 5{,}625} = \sqrt{33.5} \approx 5.79$

Practice

1. Eight children have the ages of 5, 6, 3, 8, 2, 7, 8, 9. Calculate: a. The range; b. The mean; c. Standard deviation.

2. Ten pupils scored the following marks on a Maths test: 84, 82, 78, 65, 69, 72, 70, 88, 67, 75. Find the mean and standard deviation of their scores.

3. The ages of 8 children are 3, 4, 4, 5, 7, 8, 12, 13. Calculate the mean and standard deviation of their ages.

4. The shoe sizes of 20 pupils are shown in the table below. Calculate the mean and standard deviation of their shoe sizes.

Shoe size	35	36	37	38	39	40
Frequency (f)	2	5	6	4	2	1

5. The table below gives the weights, to the nearest half kilogramme, of the babies born in a hospital during a one-month period. Estimate the mean and standard deviation of the data, correct to 2 decimal places.

Weight (kg)	2.5	3	3.5	4	4.5
Frequency (f)	3	5	7	4	1

6. Sia is a farmer. She measures the height of her pepper plants, which are given in the table below. Calculate the mean and standard deviation of the data, correct to 2 decimal places.

Height (cm)	20-24	25-29	30-34	35-39	40-44
Frequency (f)	2	6	7	3	2

7. The table below gives the marks that 30 pupils achieved on an assignment, which was worth 15 marks in total. Calculate the mean and standard deviation, correct to 2 decimal places.

Marks	1-3	4-6	7-9	10-12	13-15
Frequency (f)	3	7	10	8	2

Lesson Title: Mean deviation	**Theme:** Probability and Statistics
Practice Activity: PHM4-L047	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to calculate the mean deviation of ungrouped and grouped data.

Overview

Mean deviation is another measure of dispersion that tells us how close data points generally are to the mean. It is similar to standard deviation. A low mean deviation indicates that points are generally close to the mean (the deviation is low). A high mean deviation indicates that points are spread out over a wide range of values, farther from the mean (the deviation is high).

Ungrouped data: $MD = \frac{\sum |x-\bar{x}|}{n}$, where x is each piece of data, $\bar{x}$ is the mean, and n is the total frequency.

Grouped data: $MD = \frac{\sum f|x-\bar{x}|}{\sum f}$, where x is each piece of data, $\bar{x}$ is the mean, and f is frequency.

We need to take several steps to solve each problem. We will use a table to organise these calculations. When drawing a table, it is important to have a column for each term that is needed in the formula.

Notice that each formula requires us to find the mean. The mean does not get a column in the table. We will calculate the mean of each data set separately.
For grouped data with class intervals, find the mid-point of each interval. The mid-points are used for x in the mean deviation formula.

Solved Examples

1. The heights of 8 pupils in centimetres are 160, 165, 163, 159, 158, 157, 162, and 164. Calculate the mean deviation of their heights.

   Solution:
   Step 1. Calculate mean: $\bar{x} = \frac{160+165+163+159+158+157+162+164}{8} = \frac{1288}{8} = 161$ centimetres.
   Step 2. Draw a table as follows:

$\boldsymbol{x}$	$\boldsymbol{x-\bar{x}}$	$\boldsymbol{\lvert x-\bar{x}\rvert}$
160	$160-161=-1$	1
165	$165-161=+4$	4
163	$163-161=+2$	2

159	$159 - 161 = -2$	2
158	$158 - 161 = -3$	3
157	$157 - 161 = -4$	4
162	$162 - 161 = +1$	1
164	$164 - 161 = +3$	3
	Total $= \sum \lvert \boldsymbol{x} - \bar{\boldsymbol{x}} \rvert =$	20

Step 3. Calculate the mean deviation:

$$MD = \frac{\sum |x-\bar{x}|}{n} = \frac{20}{8} = 2.5$$

2. A test worth 10 marks was given to 40 pupils. The results are in the table below. Calculate: a. The range; b. The mean correct to 1 decimal place; c. The mean deviation correct to 1 decimal place.

Marks	1	2	3	4	5	6	7	8	9	10
Frequency (f)	1	2	4	3	5	7	8	6	1	3

Solutions:

a. Range: $10 - 1 = 9$ marks

b. Fill the first 3 columns of the table below, and calculate mean: $\bar{x} = \frac{\sum fx}{\sum f} = \frac{239}{40} = 5.98 \approx 6$ marks

c. Fill the remaining columns of the table below, and calculate mean deviation: $\text{MD} = \frac{\sum f|x-\bar{x}|}{\sum f} = \frac{71}{40} = 1.755 \approx 1.8$

$\boldsymbol{x}$	$\boldsymbol{f}$	$\boldsymbol{fx}$	$\boldsymbol{x - \bar{x}}$	$\lvert\boldsymbol{x - \bar{x}}\rvert$	$\boldsymbol{f}\lvert\boldsymbol{x - \bar{x}}\rvert$
1	1	1	$1 - 6 = -5$	5	5
2	2	4	$2 - 6 = -4$	4	8
3	4	12	$3 - 6 = -3$	3	12
4	3	12	$4 - 6 = -2$	2	6
5	5	25	$5 - 6 = -1$	1	5
6	7	42	$6 - 6 = 0$	0	0
7	8	56	$7 - 6 = +1$	1	8
8	6	48	$8 - 6 = +2$	2	12
9	1	9	$9 - 6 = +3$	3	3
10	3	30	$10 - 6 = +4$	4	12
Totals:	$\sum f = 40$	$\sum fx = 239$			$\sum f\lvert x - \bar{x}\rvert = 71$

3. The scores of 20 pupils on a Mathematics exam are given in the table below. Calculate correct to 1 decimal place: a. The mean; b. The mean deviation.

Marks (%)	40-49	50-59	60-69	70-79	80-89	90-99
Frequency (f)	1	2	6	7	3	1

Solution:

Mean: Fill the first 4 columns of the table below, and apply the mean formula:

$\bar{x} = \frac{\sum fx}{\sum f} = \frac{1410}{20} = 70.5$ marks

Mean deviation: Fill the remaining columns of the table below, and apply the formula:

$\text{MD} = \frac{\sum f|x-\bar{x}|}{\sum f} = \frac{188}{20} = 9.4$

Interval	Mid-point (x)	f	fx	$x - \bar{x}$	$\lvert x - \bar{x}\rvert$	$f\lvert x - \bar{x}\rvert$
40-49	44.5	1	44.5	$44.5 - 70.5 = -26$	26	26
50-59	54.5	2	109	$54.5 - 70.5 = -16$	16	32
60-69	64.5	6	387	$64.5 - 70.5 = -6.0$	6	36
70-79	74.5	7	521.5	$74.5 - 70.5 = +4$	4	28
80-89	84.5	3	253.5	$84.5 - 70.5 = +14$	14	42
90-99	94.5	1	94.5	$94.5 - 70.5 = +24$	24	24
	Totals:	$\sum f = 20$	$\sum fx = 1410$			$\sum f\lvert x - \bar{x}\rvert = 188$

Practice

1. Eight children have ages 5, 6, 3, 8, 2, 7, 8, 9. Calculate the mean deviation.
2. The shoe sizes of 10 football players are 39, 41, 42, 40, 38, 37, 40, 41, 40, and 42. Calculate the mean and mean deviation.
3. The weights of the same 10 football players in kilogrammes are 67, 75, 72, 85, 80, 71, 83, 89, 76, and 72. Calculate: a. The range; b. The mean; c. The mean deviation.
4. Hawa raises chickens on her farm. The weights of 6 of her chickens in kilogrammes are 0.8, 1.5, 1.3, 1.0, 1.2 and 0.8. Find the mean and mean deviation of their weights, correct to 1 decimal place.
5. The ages of 15 children are given in the table below. Calculate the mean and mean deviation of their ages, correct to 2 decimal places.

Age (years)	2	3	4	5	6
Frequency (f)	1	1	5	3	5

6. The heights of 20 pupils in centimetres are given in the table. Calculate the mean and mean deviation correct to 3 significant figures.

Height (cm)	154	155	156	157	158	159
Frequency (f)	4	6	3	2	3	2

7. The weights in kilogrammes of 19 babies are given in the table. Calculate the mean and mean deviation.

Weight (kg)	2.5	3	3.5	4	4.5	5
Frequency (f)	2	5	6	4	1	1

8. The heights of 20 children are given in the table below, to the nearest 5 cm. Find the mean and mean deviation of their heights.

Height (cm)	120-124	125-129	130-134	135-139
Frequency (f)	3	6	7	4

9. The ages of 15 Maths teachers are given in the table below. Find the mean deviation.

Age (years)	31-35	36-40	41-45	46-50	51-55
Frequency (f)	2	1	4	5	3

10. The weights of 12 football players are given in the table below. Find the mean and mean deviation.

Weight (kg)	65-69	70-74	75-79	80-84	85-89
Frequency (f)	1	3	4	3	1

Lesson Title: Statistics problem solving	**Theme:** Probability and Statistics
Practice Activity: PHM4-L048	**Class:** SSS 4

Learning Outcome
By the end of the lesson, you will be able to solve advanced problems involving statistics.

Overview

In this lesson, you will use information from previous lessons to solve various statistics problems, including those with probability questions. The WASSCE exam often features problems with both statistics and probability questions.

Solved Examples

1. The frequency distribution shows the marks scored by 30 pupils on a Maths exam.

Marks (%)	30-39	40-49	50-59	60-69	70-79	80-89	90-99
Frequency	2	4	5	8	7	3	1

a. Draw a cumulative frequency curve for the distribution.
b. Use the graph to find the 45^{th} percentile.
c. If students must score more than 68% to pass, use the graph to find the probability that a student chosen at random passed the test.

Solutions:

a. Before drawing the cumulative frequency curve, complete a cumulative frequency table:

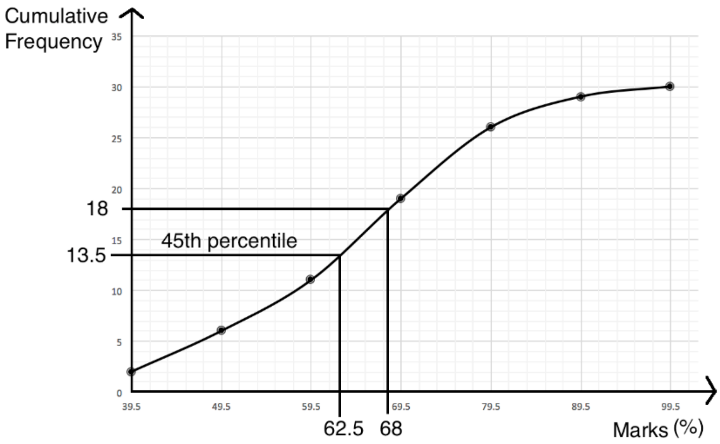
Pupils' Marks			
Marks	**Frequency**	**Upper Class Boundary**	**Cumulative Frequency**
30 – 39	2	39.5	2
40 – 49	4	49.5	2+4=6
50 – 59	5	59.5	6+5=11
60 – 69	8	69.5	11+8=19
70 – 79	7	79.5	19+7=26
80 – 89	3	89.5	26+3=29
90 – 99	1	99.5	29+1=30
Total	30		

See curve below.

b. Find the position of the 45^{th} percentile: $\frac{n}{100}\sum f = \frac{45}{100}(30) = \frac{1350}{100} = 13.5$
Identify the 45^{th} percentile on the curve as 62.5 marks. (see curve below).

c. To identify the number of pupils scoring above 68%, first identify 68 marks on the c.f. curve. 68 marks corresponds to a cumulative frequency of 18. If 18 pupils scored 68 or lower, then the number that passed is $30 - 18 = 12$.
Probability that a student passed $= \frac{\text{passing students}}{\text{all students}} = \frac{12}{30} = 0.4$.

Curve:



2. The weights of 8 babies in kilogrammes are: 3.5, 5.0, 5.5, 3.0, 4.0, 3.5, 4.5, and 3.0. Calculate:
   a. The mean;
   b. Standard deviation;
   c. The probability that a baby chosen at random will weigh at least 4 kilogrammes.

Solutions:

a. Mean: $\bar{x} = \frac{3.5+5+5.5+3.0+4.0+3.5+4.5+3.0}{8} = \frac{32}{8} = 4$ kg

b. Standard deviation: $s = \sqrt{\frac{1}{n}\sum(x-\bar{x})^2} = \sqrt{\frac{6}{8}} = \sqrt{0.75} \approx 0.87$

$\boldsymbol{x}$	$\boldsymbol{x - \bar{x}}$	$\boldsymbol{(x - \bar{x})^2}$
3.5	$3.5 - 4 = -0.5$	0.25
5.0	$5 - 4 = +1$	1
5.5	$5.5 - 4 = +1.5$	2.25
3.0	$3 - 4 = -1$	1
4.0	$4 - 4 = 0$	0
3.5	$3.5 - 4 = -0.5$	0.25
4.5	$4.5 - 4 = +0.5$	0.25
3.0	$3 - 4 = -1$	1
Total	$= \sum(x-\bar{x})^2 =$	6

c. Probability $= \frac{\text{babies 4.0 kg and heavier}}{\text{all babies}} = \frac{4}{8} = 0.5$

3. The scores of 50 pupils on a test are shown in the frequency table.

Marks (%)	20-29	30-39	40-49	50-59	60-69	70-79	80-89	90-99
Frequency	2	3	5	4	9	13	11	3

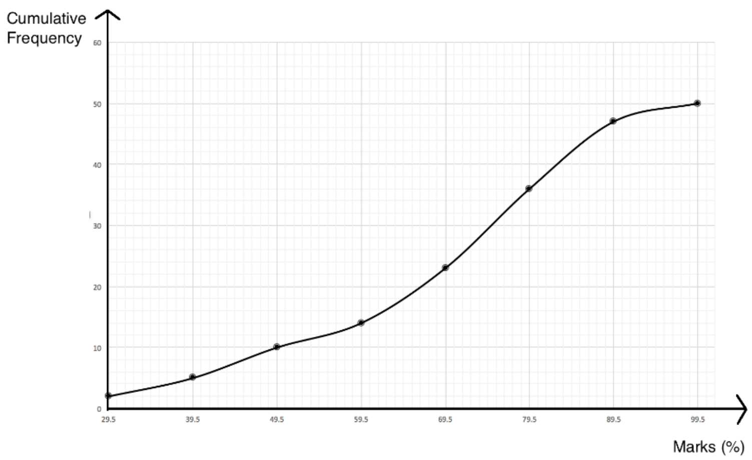
a. Draw a cumulative frequency table.
b. Use the table to draw the cumulative frequency curve.
c. If 70% of pupils passed, find the pass mark.
d. If 8% of candidates were awarded distinction, estimate the lowest mark for distinction.
e. How many pupils were awarded distinction?

Solutions:

a. Cumulative frequency table:

Pupils' Scores on a Test			
Marks	**Frequency**	**Upper Class Boundary**	**Cumulative Frequency**
20-29	2	29.5	2
30-39	3	39.5	2+3=5
40-49	5	49.5	5+5=10
50-59	4	59.5	10+4=14
60-69	9	69.5	14+9=23
70-79	13	79.5	23+13=36
80-89	11	89.5	36+11=47
90-99	3	99.5	47+3=50
Total	50		

b. Cumulative frequency curve:



c. If 70% of pupils passed, then 30% of pupils failed. This question is just another way of asking you to find the 30th percentile.

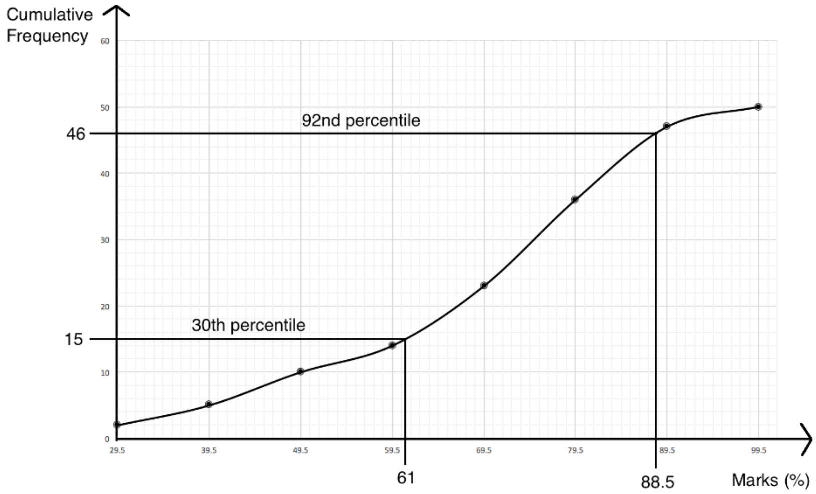
Find the position of the 30th percentile: $\frac{n}{100}\sum f = \frac{30}{100}(50) = \frac{1500}{100} = 15$

The 30th percentile is 61%, which is the pass mark.

d. If 8% were awarded distinction, then 92% were not. Find the position of the 92nd percentile: $\frac{n}{100}\sum f = \frac{92}{100}(50) = \frac{4600}{100} = 46$

The 92^{nd} percentile is 88.5%, which is the lowest mark for distinction.

e. Four pupils were awarded distinction. The position of the 92^{nd} percentile is 46, which means there are 4 pupils above distinction ($50 - 46 = 4$).



Practice

1. The following are the ages of 20 football players: 25, 22, 23, 26, 27, 23, 24, 24, 25, 26, 27, 26, 22, 24, 25, 21, 22, 23, 25, 27.
    a. Construct a frequency table for the grouped data, without class intervals.
    b. Calculate, correct to 2 decimal places: i. The mean; ii. Standard deviation.
    c. Find the probability that a football player selected at random is under 25 years old.
2. The table below gives the scores of 25 pupils on an exam.

Score	0-4	5-9	10-14	15-19
Frequency (f)	2	6	7	5

Calculate:

    a. The mean score.
    b. The mean deviation.
    c. The probability that a pupil selected at random scored 10 or more points.
3. Martin is a goat farmer. The weights of his 20 goats are given in the table below, to the nearest 5 kilograms. Calculate, correct to 1 decimal place: a. The mean; b. The mean deviation.

Weight (kg)	30	35	40	45	50
Frequency (f)	2	5	7	4	2

4. The scores of 12 pupils on an exam are given in the table.

Score	51-60	61-70	71-80	81-90	91-100
Frequency (f)	1	2	4	4	1

    a. Calculate the mean score.
    b. Calculate the standard deviation of the distribution.
    c. If the pass mark is 71, what percentage of pupils passed?
    d. Calculate the probability that a pupil selected at random failed the exam.

Answer Key – Term 1

Lesson Title: Basic numeration
Practice Activity: PHM4-L001

1. $6\frac{11}{15}$ 2. $7\frac{1}{6}$ 3. 14.09 4. 38.0 5. 0.0054
6. 4.167 7. 1.25% 8. 1% 9. 4% 10. 40 cm

Lesson Title: Sequences
Practice Activity: PHM4-L002

1. 45 2. -27 3. 19 4. $U_3 = 35$ 5. 119
6. $6{,}561$ 7. $\frac{256}{2{,}187}$ 8. 9 9. $U_n = 5(2)^{n-1}$

Lesson Title: Series
Practice Activity: PHM4-L003

1. 140 2. -24 3. 150 4. 121 5. $-2{,}046$
6. $\frac{129}{8}$ or $16\frac{1}{8}$ 7. a. 5; b. 3; c. 153

Lesson Title: Problem solving using sequences and series
Practice Activity: PHM4-L004

1. Le $680{,}000.00$ 2. a. Le $23{,}000.00$; b. Le $192{,}000.00$
3. a. 15 people; b. 175 people 4. a. 16 bricks; b. 127 bricks
5. 5 groups

Lesson Title: Ratios
Practice Activity: PHM4-L005

1. a. L = 16 cm, w = 15 cm; b. 5:6
2. a. 140 kg; b. Le4,400.00; c. 4.32 km; d. 210 mangoes
3. a. 48 m; b. 40 min; c. 3.5 litres; d. 45 pupils
4. a. 8 : 17 $>$ 7 : 15; b. 13 : 7 $>$ 11 : 6; c. 7 g : 8 g $>$ 11 m : 13 m;
   d. Le 170.00 : Le 90.00 $>$ Le 300.00 : Le 160.00

Lesson Title:	Rates
Practice Activity:	PHM4-L006

1. 450 litres
2. 8 labourers
3. a. 12 hours; b. 2 hours
4. Le 486,000.00
5. 37 bicycles/day
6. a. Le 10,000.00; b. Le 18,000.00
6. Le 3,800.00.00
7. 68 km/h
7. 32 cm
8. 1,000 people

Lesson Title:	Proportional division
Practice Activity:	PHM4-L007

1. 81
2. 500
3. 240
4. Momodu = Le 1,000.00 Musa = Le 400.00
5. Le 3,712,000.00
6. Fatu = Le 120,000.00 Adama = Le 180,000.00
7. Abdul = Le 560,000.00 Karim = Le 640,000.00 Unisa = Le 800,000.00

Lesson Title:	Speed
Practice Activity:	PHM4-L008

1. 90 km
2. 3.6 km/h
3. 13.3 km/h
4. 4.5 m/s
5. 6 m/s
6. 22.22 m/s
7. 180 km
8. 16 km/h

Lesson Title:	Applications of percentages – Part 1
Practice Activity:	PHM4-T1-L009

1. a. Le 720.00; b. Le 40,800.00
2. Le 730,000.00
3. Le 235,000.00
4. a. Le 650,000.00; b. Le 550,000.00; c. Le 85,000.00; d. Le 1,115,000.00
5. 3%
6. Le 600,000.00 is the principal that should be invested, and the total amount after 3 years is Le 690,000.00.
7. Le 100,000.00
8. The interest is Le 107,753.13. The additional interest when compound interest is applied instead of simple interest is Le 7,753.13.

Lesson Title:	Applications of percentages – Part 2
Practice Activity:	PHM4-L010

1. a. Profit of Le 44,000.00; b. Le 24,400.00
2. a. Le 300,000.00; b. Le 200,000.00; 25%

3. Le $1{,}080{,}000.00$
4. a. Le $1{,}500{,}000.00$; b. Le $3{,}000{,}000.00$
5. a. Le $168{,}750.00$; b. Le $375{,}000.00$
6. a. Le $36{,}720.00$; b. 23.5%

Lesson Title:	Applications of percentages – Part 3
Practice Activity:	PHM4-L011

1. Le $850{,}000.00$
2. a. Le $1{,}496{,}000.00$; b. 40%
3. a. Le $5{,}825{,}000.00$**;** b. Le $1{,}825{,}000.00$
4. Le $150{,}000.00$
5. Le $60{,}000.00$
6. She needs ₦$480{,}769.23$ for Nigeria, £$1{,}020.41$ for GB and €$1{,}123.60$ for Germany

Lesson Title:	Indices
Practice Activity:	PHM4-L012

1. $5a^3b^6$ 2. $\frac{3x^3}{y}$ 3. 9 4. 16
5. 4 6. $2\frac{1}{15}$ 7. 2 8. -10

Lesson Title:	Logarithms – Part 1
Practice Activity:	PHM4-L013

1. 3 2. – 2 3. $\frac{3}{2}$ 4. – 5 5. 32
6. a. 1.5798; b. 0.6668; c. 1.5160
7. a. 17.63 b. 1546 c. 659.7
8. 138.6
9. a. $\bar{1}.0899$ b. $\bar{2}.9943$ c. $\bar{3}.8156$
10. a. 0.02026; b. 0.0001023 c. 0.1589

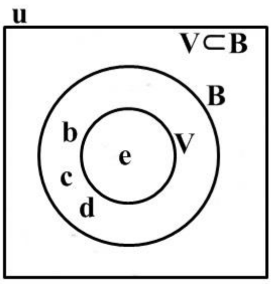
Lesson Title:	Logarithms – Part 2
Practice Activity:	PHM4-L014

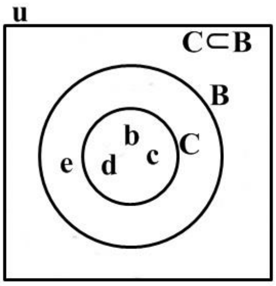
1. a. $\log_2 48$, b. $\log_5 24$, c. $\log_4 128$, d. $\log_3 36$
2. a. 1.4771, b. 1.7481, c. 1.8451
3. a. 1; b. $\log_3 \frac{16}{9}$
4. a. 1.18; b. -0.318; c. -0.636

5. a. 1.067; b. 0.7872
6. a. $8\log_3 x$; b. 3; c. 17
7. a. 1.293; b. -0.138; c. 1.795
8. a. 2; b. $\frac{2}{3}$; c. 4
9. a. 7; b. 7
10. $a = 1, b = 0$ or $a = -2, b = 3$

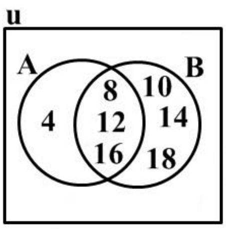
Lesson Title:	Representing sets with diagrams and symbols
Practice Activity:	PHM4-L015

1. Venn diagrams:

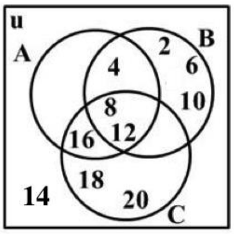




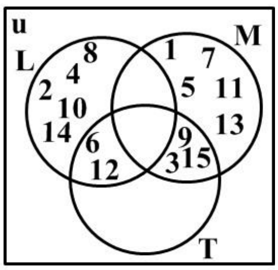
2. a. A = {4, 8, 12, 16}   B = {8, 10, 12, 14, 16, 18}; b. A ∩ B = {8, 12, 16}; c. Venn diagram:



3. a. i{4, 8, 12}; ii. {8, 12}; iii. {8, 12, 16}; iv. {8, 12}; b. Venn diagram:



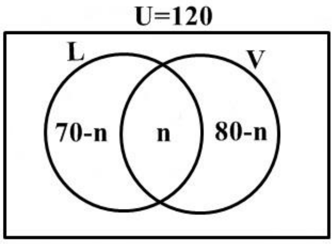
4. a. M∩L = { }, M∩T = {3, 9, 15}, L∩T = {6, 12}, M∩L∩T = { }; b. Venn diagram:



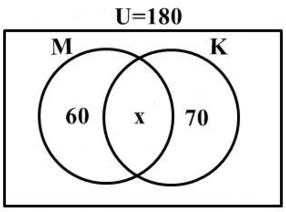
5. a. A’ = {10, 40, 50}; b. B’ = {10, 40, 60}; c. A’∩B’ = {10, 40}
6. a. U= {5, 10, 15, 20, 25, 30, 35, 40}, P = {5, 25, 30, 40}, Q = {5, 20, 35, 40}; P’ = {10, 15, 20, 35}; c. Q’ = {10, 15, 25, 30}; d. P’∩Q’ = {10, 15}

Lesson Title:	Solving problems involving sets
Practice Activity:	PHM4-L016

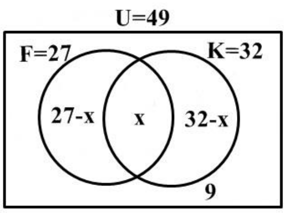
1. See Venn diagram below; b. 30 students; c. 40 played Lawn tennis only



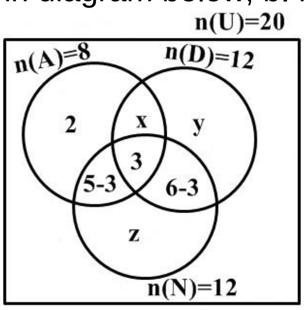
2. a. See Venn diagram below; b. i. 50   ii. 120   iii. 110



3. a. See Venn diagram below; b. i. 19   ii. 8   iii. 13



4. a. 27; b. 15
5. a. See Venn diagram below; b. i. 4; ii. 6



Lesson Title:	Operations on Surds
Practice Activity:	PHM4-L017

1. $11\sqrt{2}$ 2. $\frac{5}{2}\sqrt{7}$ 3. $9\sqrt{5}$ 4. $4\sqrt{7}$ 5. $n = 1$

6. a. 105 b. $16\sqrt{10}$ 7. a. 32 b. $144\sqrt{6}$ 8. a. $32\sqrt{3}$ b. $50\sqrt{15}$
9. a. 18 b. $48\sqrt{3}$ 10. $10\sqrt{5}$ 11. $3\sqrt{2}-4$ 12. $9\sqrt{6}-6$
13. $12\sqrt{5}-30$ 14. $24+12\sqrt{2}$ 15. $1\frac{1}{5}$ 16. $31+12\sqrt{3}$

Lesson Title: Simplifying surds
Practice Activity: PHM4-L018

1. a. $\frac{3\sqrt{2}}{4}$; b. $\sqrt{2}$; c. $\frac{\sqrt{6}}{4}$; d. $\sqrt{7}$; e. $\frac{2\sqrt{6}}{3}$
2. a. $\frac{12+3\sqrt{3}}{13}$; b. $\frac{3-\sqrt{2}}{7}$; c. $\frac{25-5\sqrt{2}}{23}$; d. $\frac{42+12\sqrt{2}}{41}$; e. $\frac{14}{13}+\frac{7}{26}\sqrt{3}$
3. a. $\frac{3}{4}\sqrt{2}-\frac{1}{2}\sqrt{6}$; b. $\frac{5}{3}+\frac{\sqrt{15}}{9}$; c. $\frac{4}{5}\sqrt{5}+\frac{\sqrt{15}}{5}$
4. $2+2\sqrt{5}$
5. $6\sqrt{10}-3\sqrt{6}$
6. a. -1; b. $-16+3\sqrt{2}$
7. $3+2\sqrt{2}$
8. $-\frac{24}{7}-\frac{13}{7}\sqrt{2}$, where $a=-\frac{24}{7}, b=-\frac{13}{7}$

Lesson Title: Simplification and factorisation
Practice Activity: PHM4-L019

1. $2x+y$ 2. $8p^2q-3pq^2$ 3. $2u-6uv$ 4. $-26m+14n$
5. m^2+m-6 6. $x^2-8x+16$ 7. n^2+6n+9 8. $8mn-2n+19m-2$
9. $3x^3-12x^2+12x$ 10. $9(ab-3)$ 11. $a(a^2+b)$ 12. $4x^2(4x^2-2x+1)$
13. $9a^2(9a+1)$ 14. $3s^2t^2(6s^3t^4-2s^2t+1)$ 15. $5uv^2(v-5+3u)$
16. $(d-e)(c-d)$ 17. $(1+5a)(p+q)$ 18. $(x+2)(x+3)$
19. $(2x-3)(5x+2)$ 20. $(2+3x)(1-2x)$

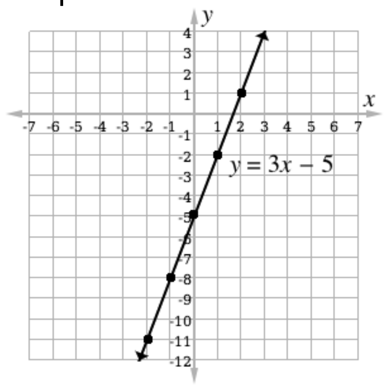
Lesson Title: Functions
Practice Activity: PHM4-L020

1. a. No; Since there are repetitions of x-values (5) with different y-values, then this relation is not one-to-one, and is therefore not a function.
   b. Yes; The relation is a function because every x-value is unique and is associated to only one value of y.
2. a. Yes; Each element of the domain gives one and only element in the range. It is okay for two or more values in the domain to share a common value in the range. That is, even though the elements 5 and 10 in the domain share the same value of 2 in the range, this relation is still a function.
   b. No; The element 15 has two arrows, which point to both 7 and 9. Therefore, this relation is not a function.
3. a. 5; b. 7; c. -3

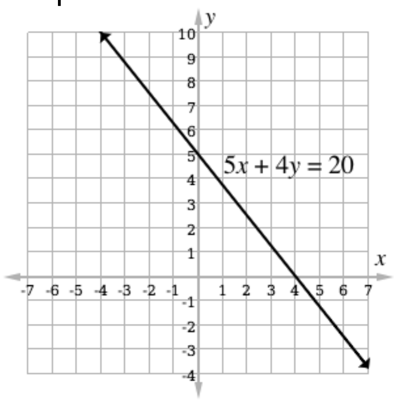
4. a. -3; b. 0; c. -4
5. $\{10, 16, 22\}$
6. $\{2, 5, 10\}$

Lesson Title:	Graphing linear functions
Practice Activity:	PHM4-L021

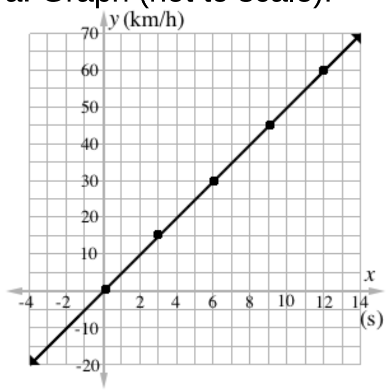
1. Graph:



2. Graph:



3. a. Graph (not to scale):

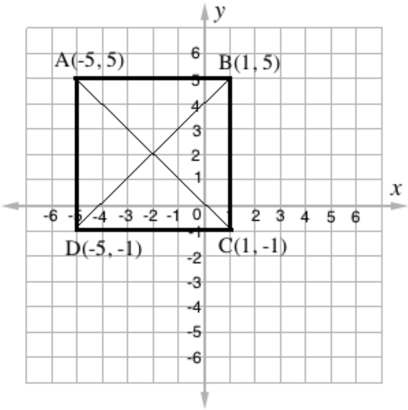


   b. i. 25 km/h; ii. 40 km/h
   c. i. 8 seconds; ii. 13 seconds
4. a. 5; b. $-\frac{2}{5}$
5. $m = 15$

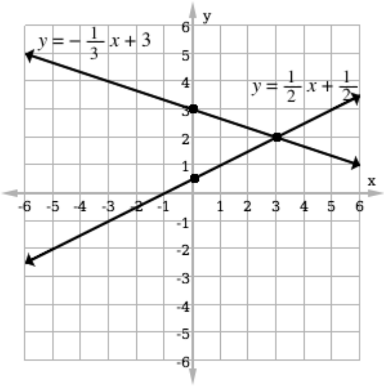
Lesson Title:	Applications of linear functions
Practice Activity:	PHM4-L022

1. $y = 3x + 2$
2. $y = -\frac{2}{3}x + \frac{11}{3}$

3. a. See graph below (not to scale); b. Linear equations: $y = x + 4$, $y = -x$



4. See graph below (not to scale); point of intersection: $(3, 2)$.

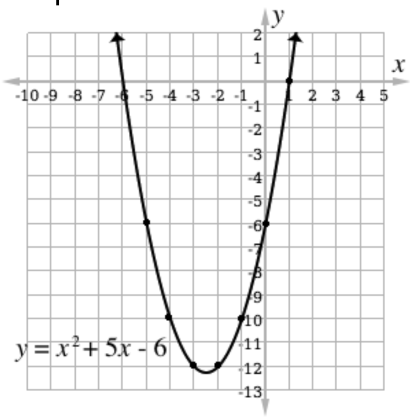


Lesson Title: Distance and mid-point formulae
Practice Activity: PHM4-L023

1. a. $3\sqrt{10}$; b. $\sqrt{2}$; c. $\sqrt{13}$; d. $p\sqrt{2}$; e. $\sqrt{2^2 + p^2}$ =$\sqrt{4 + p^2}$
2. 5
3. $PQ^2 = 80,\ QR^2 = 20,\ PR^2 = 100$
   $PQ^2 + QR^2 = PR^2$ is true
4. a. (4, 8); b. (5, -1); c. $(2p, 3q)$
5. 6

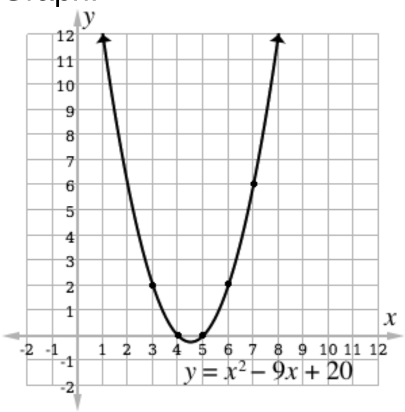
Lesson Title:	Graphing and interpreting quadratic functions
Practice Activity:	PHM4-L024

1. Graph:



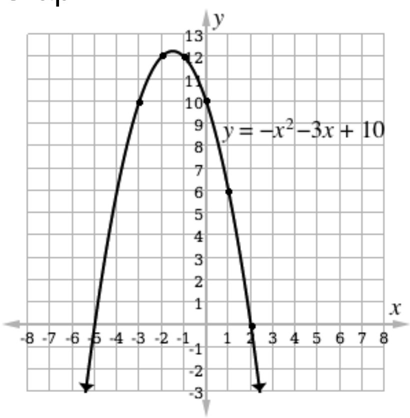
a. $x = -6,1$; b. $(-2.5, -12.25)$; c. $x = -\frac{5}{2}$ or $x = -2.5$

2. Graph:



a. $x = 4, 5$; b. $(4.5, -0.25)$; c. $y = 8.75$; d. $x = 4.5$

3. Graph:

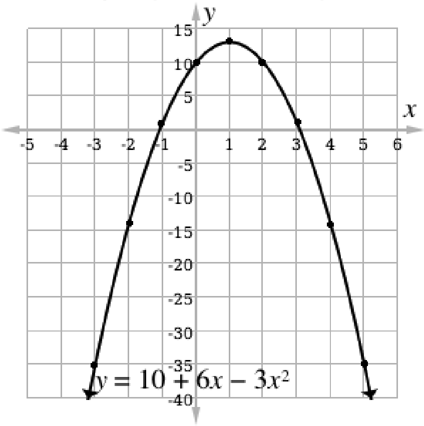


a. $x = -5, 2$; b. $(-1.5, 12.25)$; c. $x = -\frac{3}{2}$ or $x = -1.5$; d. $y = -0.7$

4. Table of values:

x	-3	-2	-1	0	1	2	3	4	5
y	-35	-14	1	10	13	10	1	-14	-35

a. Graph (not to scale):



b. $x = -1.08,\ 3.08$; c. $(1, 13)$; d. $x = 1$; e. $y = 12.25$

Lesson Title: Solving quadratic equations algebraically – Part 1
Practice Activity: PHM4-L025

1. $x = -4, 3$ 2. $x = -2, 9$ 3. $x = -4, -\frac{3}{2}$ 4. $y = b$
5. $m = 0, \frac{2}{3}$ 6. $x = \frac{1}{2}, 2$ 7. $y = -\sqrt{2}, \sqrt{2}$

Lesson Title: Solving quadratic equations algebraically – Part 2
Practice Activity: PHM4-L026

1. $\frac{4+\sqrt{10}}{3}, \frac{4-\sqrt{10}}{3}$ 2. $\frac{5+\sqrt{11}}{2}, \frac{5-\sqrt{11}}{2}$
3. a. $-\frac{2}{3}, -\frac{3}{2}$; b. $\frac{-7+\sqrt{73}}{4}, \frac{-7-\sqrt{73}}{4}$
4. a. $-1, -11$; b. $14,\ 2$; c. $6, -1$; d. $7, 2$; e. $\frac{9}{4},\ 2$

Lesson Title: Problem solving with quadratic equations
Practice Activity: PHM4-L027

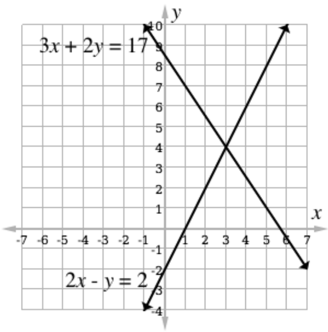
1. $-11, -13$ 2. $l = 6$ m; $w = 5$ m 3. $l = 12$ m; $w = 3$ m
4. $9, 16$ 5. $l = 16$ m; $w = 6$ m

Lesson Title: Simultaneous linear equations
Practice Activity: PHM4-L028

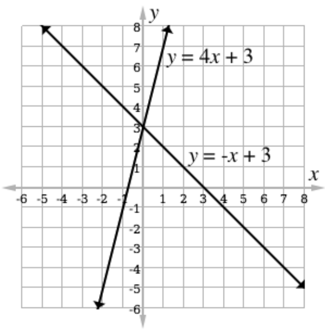
1. a. $(2, -3)$ b. $(5, 2)$

2. a. $(-2, 3)$ b. $\left(\frac{1}{2}, 2\right)$

3. a. The solution is the point of intersection, (3, 4).



b. The solution is the point of intersection, $(0, 3)$.

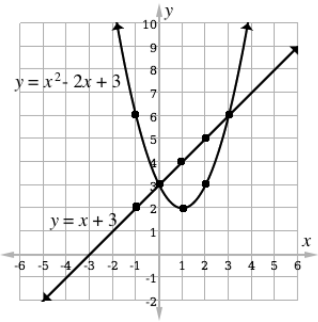


Lesson Title:	Applications of simultaneous linear equations
Practice Activity:	PHM4-L029

1. Son is 12 years, father is 34 years
2. Son is 8 years, woman is 32 years
3. 75
4. One pineapple costs Le 6,800.00, one mango costs Le 1,400.00.
5. Speed = 80 km/hour, distance = 160 km
6. Daughter will be 35 years, and father will be 50 years
7. An apple costs Le 3,000.00, a tin of milk costs Le 5,000.00
8. $a = -2, b = 3$
9. $m = 5, c = -10$
10. $a = 2, b = -1$

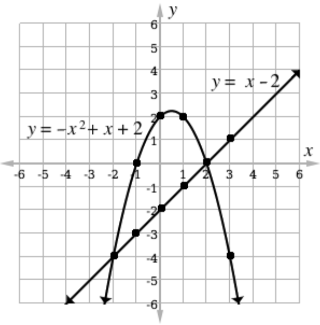
Lesson Title:	Simultaneous quadratic and linear equations
Practice Activity:	PHM4-L030

1. a. $(-3,1)(-2,-1)$; b. $(-9, 90)$, $(-1, 10)$; c. One solution: $(l, m) = (1, \frac{2}{3})$;
2. a. Graph:



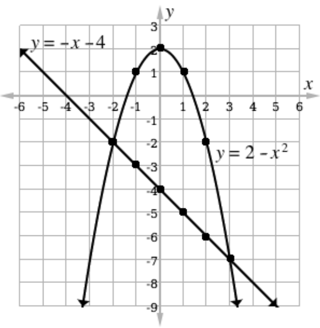
Solutions: $(0, 3)$ and $(3, 6)$

b. Graph:



Solutions: $(-2, -4)$ and $(2, 0)$

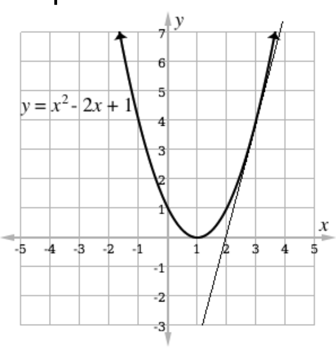
c. Graph:



Solutions: $(-2, -2)$ and $(3, -7)$

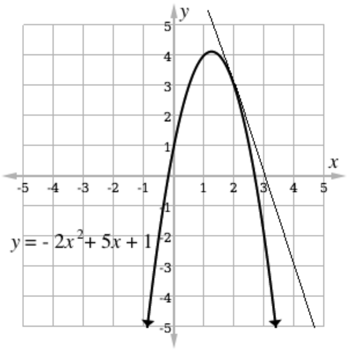
Lesson Title:	Tangent to a quadratic function
Practice Activity:	PHM4-L031

1. Graph:



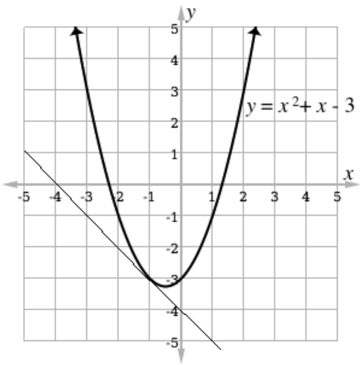
Using points $(3, 4)$ and $(2, 0)$ on the tangent line, we can find the gradient of the curve at $x = 3$ is $m = 4$.

2. Graph:



Using points $(2, 3)$ and $(3, 0)$ on the tangent line, we can find the gradient of the curve at $x = 2$ is $m = -3$.

3. Graph:

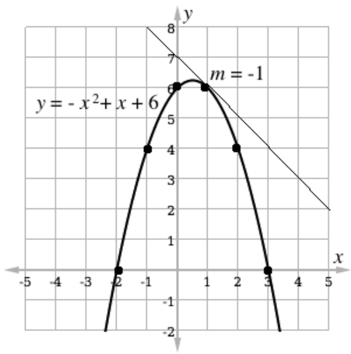


Using points $(-1, -3)$ and $(0, -4)$ on the tangent line, we can find the gradient of the curve at $x = -1$ is $m = -1$.

4. a. Completed table:

x	-2	-1	0	1	2	3
y	0	4	6	6	4	0

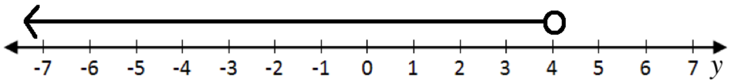
b. Graph:



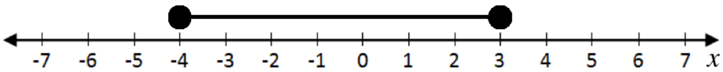
c. See above for graph of the tangent line; the slope is $m = -1$.

Lesson Title:	Inequalities
Practice Activity:	PHM4-L032

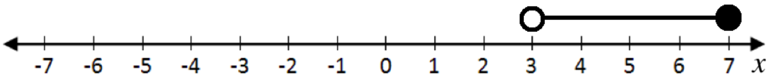
1. $y < 4$; number line:



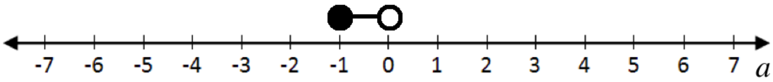
2. $-4 \leq x \leq 3$; number line:



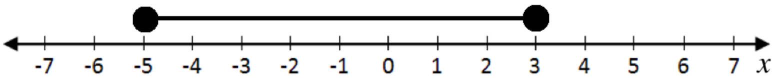
3. $3 < x \leq 7$; number line:



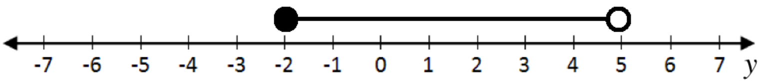
4. $-1 \leq a < 0$; number line:



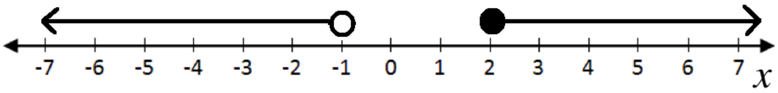
5. $-5 \leq x \leq 3$; number line:



6. $-2 \leq y < 5$; number line:



7. $x < -1$ or $x \geq 2$; number line:



Lesson Title:	Variation
Practice Activity:	PHM4-L033

1. a. $T = 4U(V-4)^2$; b. 2 or 6
2. a. $P = \frac{2F}{A}$; b. $375N$
3. a. $C = 200{,}000 + 100{,}000n$; b. $\text{Le } 1{,}400{,}000.00$; c. 14 rooms
4. a. $R = \frac{(0.02)l}{r^2}$; b. 16Ω
5. a. $F = \frac{3m_1m_2}{s^2}$; b. 90 kg
6. a. $R = -850 + 3U^2$; $6{,}650N$

Lesson Title:	Simplification of algebraic fractions
Practice Activity:	PHM4-L034

1. $\frac{3bc}{4a}$ 2. $\frac{2}{x}$ 3. $\frac{x-6}{x-7}$ 4. $\frac{3x+2}{x+3}$ 5. $-\frac{m+1}{m+2}$ 6. $-\frac{3m-1}{3m+1}$

Lesson Title:	Operations on algebraic fractions
Practice Activity:	PHM4-L035

1. $\frac{x-1}{(x-2)(x+2)}$ 2. $\frac{x^2}{(x-1)(x+1)}$ 3. $41-6y$ 4. $\frac{10x^2-3x-3}{10(x-1)(x+1)}$
5. $\frac{(x-1)(x+7)}{4x}$ 6. $\frac{y-5}{y(y-3)}$ 7. $\frac{6a}{a-5}$ 8. $\frac{2x(1+3x)}{(1-x)(1+x)}$ 9. $\frac{1}{6ab}$

Lesson Title:	Logical reasoning – Part 1
Practice Activity:	PHM4-L036

1. a. If John has measles, then he is in the hospital. $(E \Longrightarrow F)$; b. If Amid plays football, then he scores many goals. $(A \Longrightarrow B)$
2. For two sub-statements of Y:
   S: David studied very hard.
   P: David passed his Mathematics exam.
   Symbols: $\mathrm{Y} = \mathrm{S} \wedge \mathrm{P}$

   Truth table:

S	***P***	***S∧P***
T	T	T
T	F	F
F	T	F
F	F	F

3. a. The truth table for the disjunction:

A	***B***	***A* ∨ *B***
T	T	T
T	F	T
F	T	T
F	F	F

   b. The truth table for the conjunction:

A	***B***	***A∧B***
T	T	T
T	F	F
F	T	F
F	F	F

4. a. If Foday does not have a lot to eat, then he does not harvest his farm.
   b. If Sia does not live in Sierra Leone, then she does not live in Bo.
   c. If Issa is not African, then he is not from Sierra Leone.
5. a. The statements are equivalent. $A \Leftrightarrow B$

b. The statements are not equivalent. $S \Rightarrow T$, but the converse is not true because "a shape has 4 sides" does not imply that the shape is a square. It could be another type of quadrilateral.
c. The statements are equivalent. All shapes with 5 sides are pentagons, and all pentagons have 5 sides. $X \Leftrightarrow Y$

Lesson Title:	Logical reasoning – Part 2
Practice Activity:	PHM4-L037

1.

$X \Rightarrow Y$	If John practises solving Mathematics problems every day, then he is very good in Mathematics.
$Y \Rightarrow Z$	If John is a very good mathematician, then he can solve every problem in Mathematics.
$X \Rightarrow Z$	If John practises solving Mathematics problems every day, then he can solve every problem in Mathematics.
$\sim Y \Rightarrow \sim X$	If John is not a very good mathematician, then he does not practice solving Mathematics problems every day.
$\sim Z \Rightarrow \sim Y$	If John cannot solve every problem in Mathematics, then he is not a very good mathematician.
$\sim Z \Rightarrow \sim X$	If John cannot solve every problem in Mathematics, then he does not practise solving Mathematics problems every day.

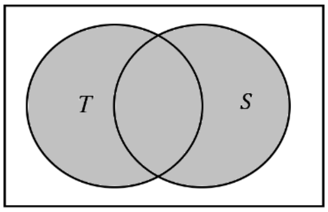
2.

a.

Truth Table:

T	S	$T \vee S$
T	T	T
T	F	T
F	T	T
F	F	F

Venn Diagram:

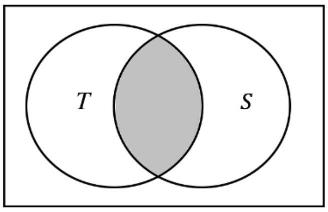


b.

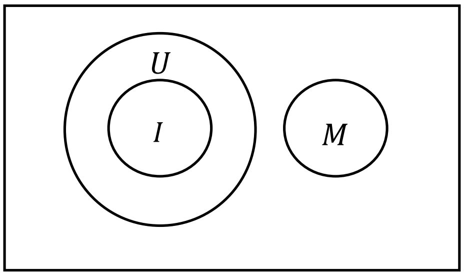
Truth Table:

T	S	$T \wedge S$
T	T	T
T	F	F
F	T	F
F	F	F

Venn Diagram:

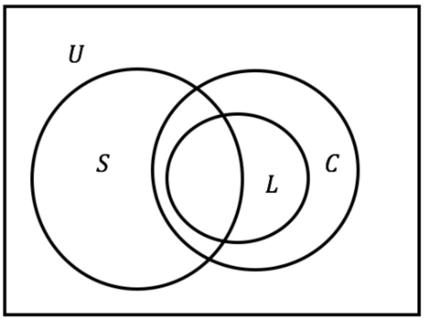


3. Venn diagram:



4.

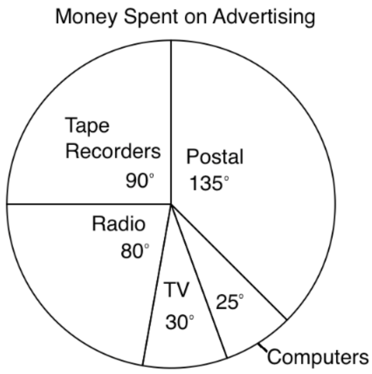
a. Illustration on a Venn diagram. Let U={all students}, S={strong students}, L={lazy students} and C={careless students}



b. i. Not valid
ii. Valid
iii. Not valid
iv. Not valid
v. Valid
vi. Not valid

Lesson Title:	Pie charts and bar charts
Practice Activity:	PHM4-L038

1. a. 600 b. 8.3% c. 160 d. Monday
2. a. $90°$
   b. Pie chart:



   c. i. $\frac{3}{8}$; ii. $\frac{1}{4}$
   d. i. $22\frac{2}{9}\%$ or 22.22%; ii. $6\frac{17}{18}\%$ or 6.94%
   e. i. Le $540{,}000.00$; ii. Le $45{,}000.00$

Lesson Title:	Mean, median, and mode of ungrouped data
Practice Activity:	PHM4-L039

1. a.

Grades (x)	Tally	Frequency (f)
45	||||	4
50	~~||||~~ |	6
55	~~||||~~ ||||	9
60	~~||||~~ |	6
65	|||	3
70	||	2

b. 55.67

c. median: 55; mode: 55

2. a. $x = 5$

b. 50 days

c. Median= 14; Mode= 14 and 16

3. a. 30 accidents

b. March

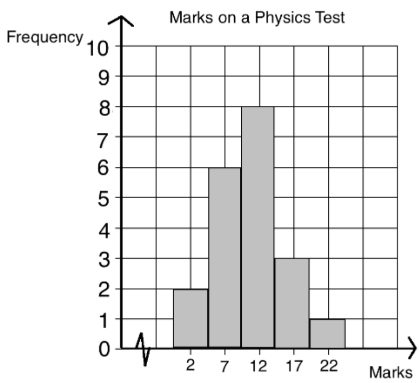
c. 6 accidents

Lesson Title:	Histograms
Practice Activity:	PHM4-L040

1. a. Frequency table:

Mark	Frequency
$0 - 4$	2
$5 - 9$	6
$10 - 14$	8
$15 - 19$	3
$20 - 24$	1

b. Histogram:



c. $10 - 14$

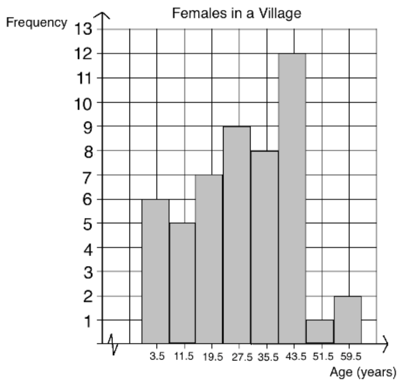
d. 60%

e. $\frac{4}{10} = 0.4$

2. a. Frequency table:

Age	Frequency
$0-7$	6
$8-15$	5
$16-23$	7
$24-31$	9
$32-39$	8
$40-47$	12
$48-55$	1
$56-63$	2

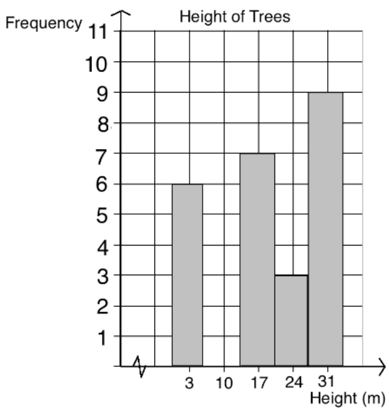
b. Histogram:



c. $40-47$

d. 36%

3. a.



b. 25 trees

c. $28-34$

d. 52%

e. $\frac{12}{25}=0.48$

4. a. 40 students

b. $4-7$

c. 20 students

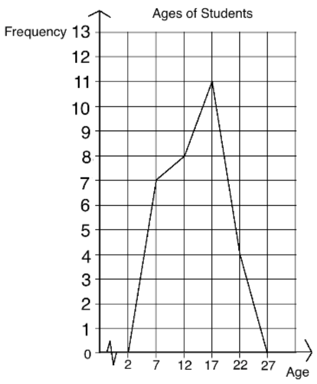
d. 25%

Lesson Title:	Frequency polygons
Practice Activity:	PHM4-L041

1. a.

Age	Frequency
$0-4$	0
$5-9$	6
$10-14$	8
$15-19$	12
$20-24$	4

b. Frequency polygon:

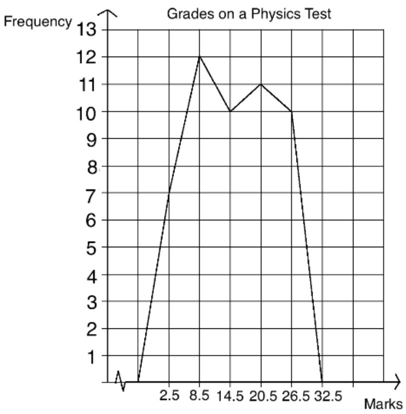


c. Modal class: $15-19$, Median class: $15-19$

2. a.

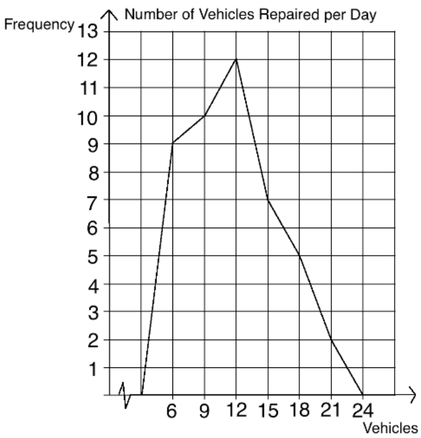
Mark	Frequency
$0-5$	7
$6-11$	12
$12-17$	10
$18-23$	11
$24-29$	10

b. Frequency polygon:



c. Modal class: $18 - 23$, Median class: $12 - 17$

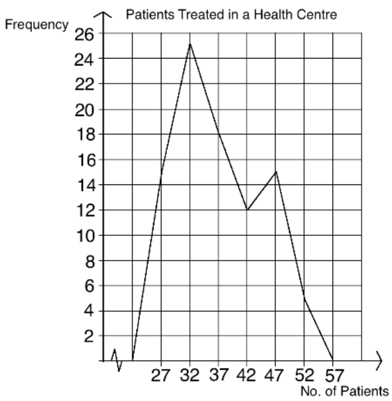
3. a. Frequency polygon:



b. Modal class: $11 - 13$, Median class: $11 - 13$

c. 31.1%

4. a. Frequency polygon:

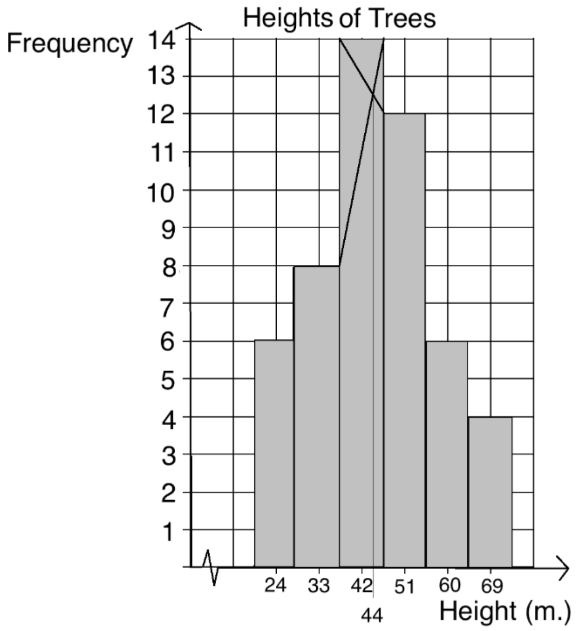


b. $30 - 34$
c. $35 - 39$

d. 16.67%

Lesson Title:	Mean, median, and mode of grouped data
Practice Activity:	PHM4-L042

1. a. 14.3; b. 15.7
2. Mean: 44.45 kg; Median: 45.43 kg
3. Note that the class intervals are: 20-28, 29-37, 38-46, and so on.
   Mean: 44.88 m; Median: 45.07 m; Mode: 44 m (see histogram below)

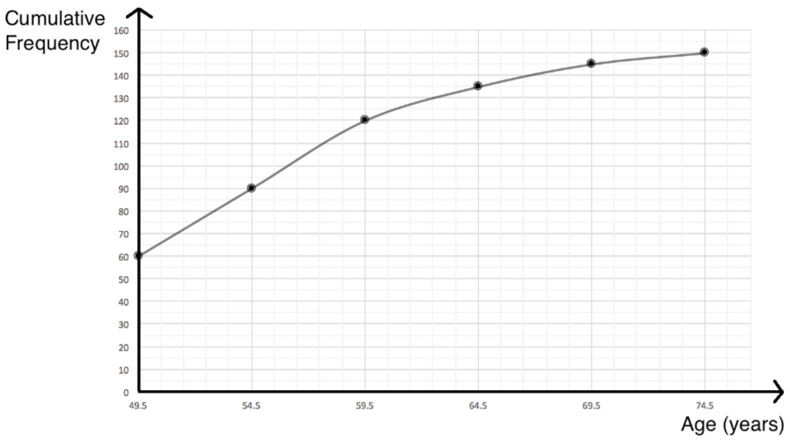


Lesson Title:	Cumulative frequency curves and quartiles
Practice Activity:	PHM4-L043

1. a. Cumulative frequency table:

Age (years)	**Cumulative frequency**
49.5	60
54.5	95
59.5	120
64.5	135
69.5	145
74.5	150

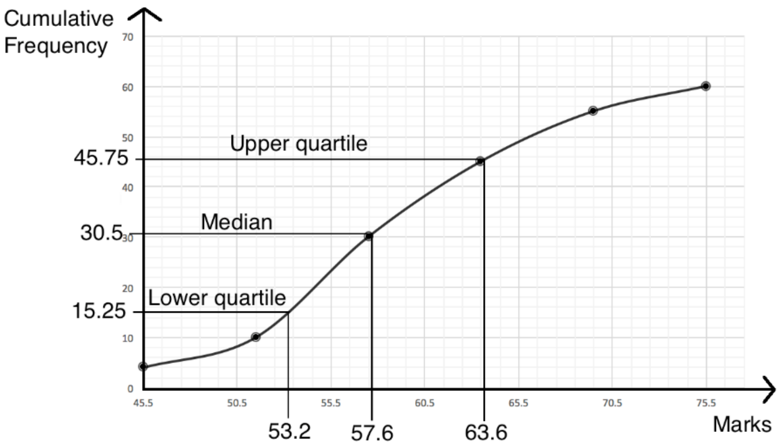
b. Cumulative frequency curve:



2. a. Cumulative frequency table:

Marks	Cumulative frequency
45.5	4
51.5	10
57.5	30
63.5	45
69.5	55
75.5	60

Cumulative frequency curve (with estimated quartiles shown):



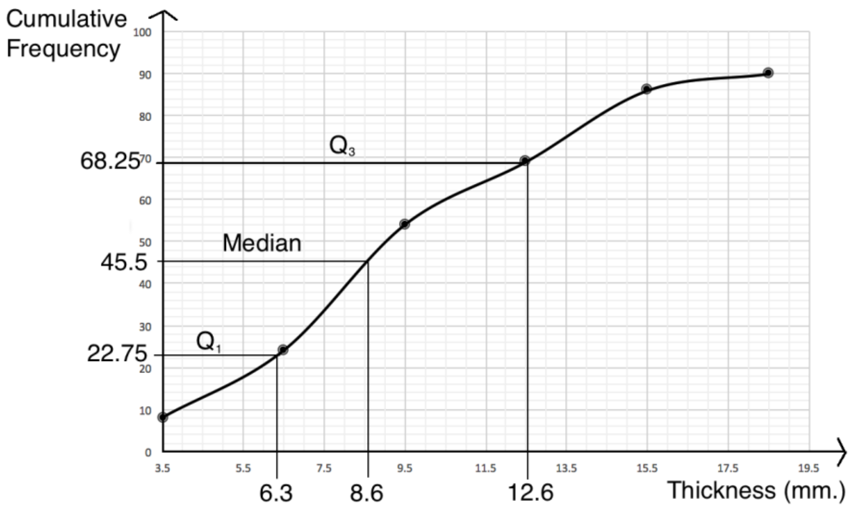
b. i. $Q_1 = 53.2$; $Q_3 = 63.6$; ii. Median $= 57.6$; iii. Interquartile range $= 10.4$

3. a. Cumulative frequency table:

Thickness less than	Cumulative frequency
3.5	32
6.5	68
9.5	68

12.5	80
15.5	96
18.5	100

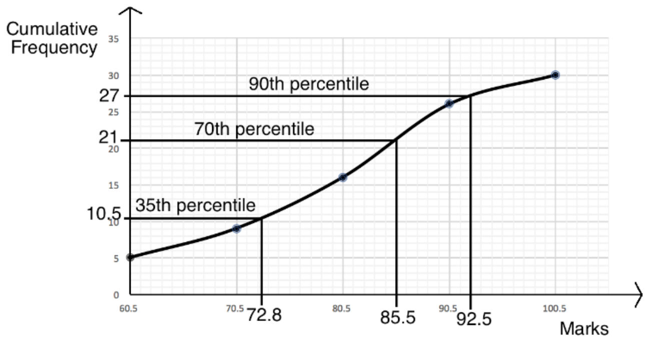
Cumulative frequency curve (with estimated quartiles shown):



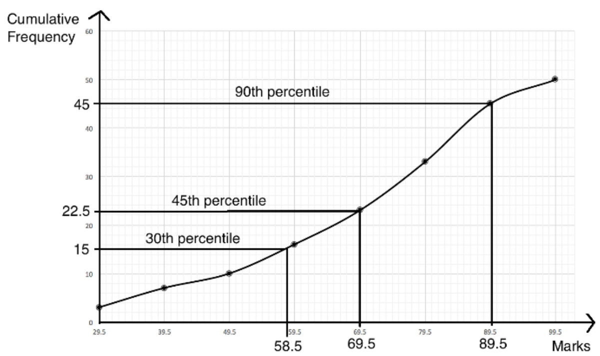
b. i. $Q_1 = 6.3$ mm; $Q_3 = 12.6$ mm; ii. Median $= 8.6$ mm;
iii. Semi-interquartile range $= 3.15$

Lesson Title:	Percentiles
Practice Activity:	PHM4-L044

1. Estimated values: a. 72.8; b. 85.5; c. 92.5. See graph:



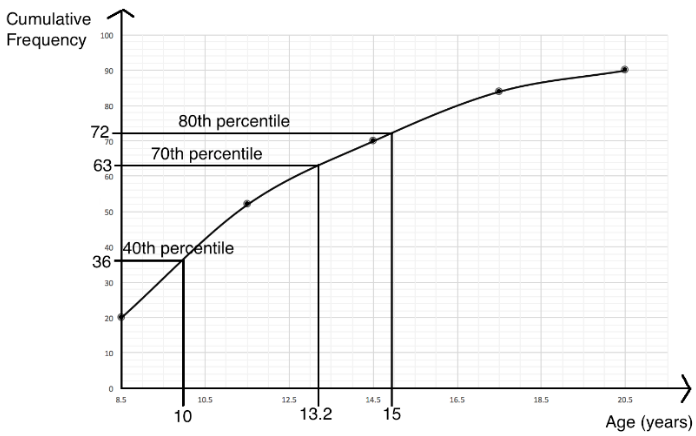
2. Estimated values: a. 58.5; b. 69.5; c. 89.5. See graph:



3. a. Cumulative frequency table:

Pupils' ages			
Age (years)	**Frequency**	**Upper Class Boundary**	**Cumulative Frequency**
5 – 8	20	8.5	20
9 – 11	32	11.5	20+32=52
12 – 14	18	14.5	52+18=70
15 – 17	14	17.5	70+14=84
18 – 20	6	20.5	84+6=90
Total	90		

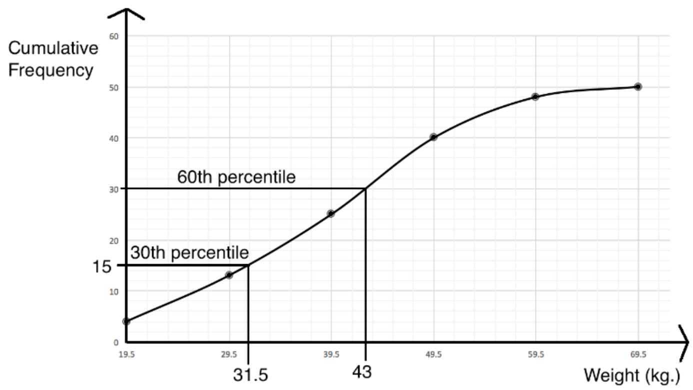
a. Cumulative frequency curve: see below.
b. 80^{th} percentile (see curve below): 15 years old
c. 36 pupils
d. 10 years old
e. 27 pupils
f. 13 years old



4. a. Cumulative frequency table:

Chimpanzees' weights			
Age (years)	**Frequency**	**Upper Class Boundary**	**Cumulative Frequency**
10 – 19	4	19.5	4
20 – 29	9	29.5	4+9=13
30 – 39	12	39.5	13+12=25
40 – 49	15	49.5	25+15=40
50 – 59	8	59.5	40+8=48
60 – 69	2	69.5	48+2=50
Total	50		

a. Cumulative frequency curve: see below.
b. 15 chimpanzees
c. 20 chimpanzees
d. 43 kg



Lesson Title:	Measures of dispersion
Practice Activity:	PHM4-L045

1. a. 10 cm; b. 125 cm; c. 11.33
2. Range: 20 marks; variance: 46.6
3. a. 15 kg; b. 27 kg; c. 21.5

Lesson Title:	Standard deviation
Practice Activity:	PHM4-L046

1. a. 7 years; b. 6 years old; c. 2.35
2. $mean \approx 75$ marks, $s \approx 7.36$
3. $mean \approx 7$ years, $s \approx 3.54$
4. $\bar{x} = 37.1$; $s = 1.3$
5. $\bar{x} = 3.38$ kg.; $s = 0.51$
6. $\bar{x} = 31.25$ cm; $s \approx 5.54$
7. $\bar{x} = 7.90$ marks; $s = 3.24$

Lesson Title:	Mean deviation
Practice Activity:	PHM4-L047

1. $MD = 2$
2. $\bar{x} = 40$; $MD = 1.2$
3. a. 22 kg; b. $\bar{x} = 77$ kg; c. $MD = 5.8$
4. $\bar{x} = 1.1$ kg; $MD = 0.2$
5. $\bar{x} = 4.67, MD = 1.02$
6. $\bar{x} = 156$ cm; $MD = 1.4$
7. $\bar{x} = 3.5$ kg; c. $MD = 0.47$
8. $\bar{x} = 130$ cm; $\mathrm{MD} = 4.2$
9. $\mathrm{MD} = 5.2$
10. $\bar{x} = 77$ kg; $\mathrm{MD} = 4.17$

Lesson Title:	Statistics problem solving
Practice Activity:	PHM4-L048

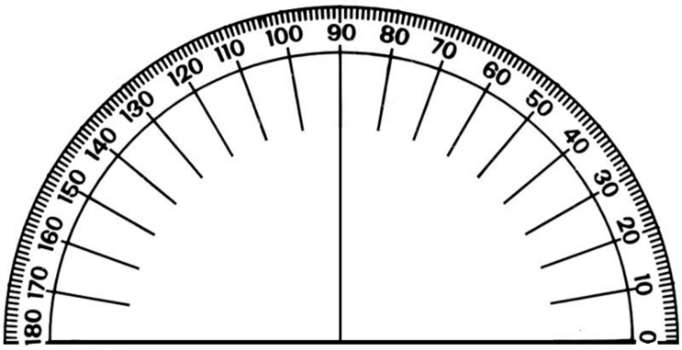
1. a. Frequency table:

Age (years)	21	22	23	24	25	26	27
Frequency (f)	1	3	3	3	4	3	3

   b. $\bar{x} = 24.35$; $s = 1.80$; c. 0.5
2. a. $\bar{x} = 10.75$; b. $MD = 4$; c. 0.6
3. a. $\bar{x} = 39.8$ kg; b. $MD = 4.3$
4. a. $\bar{x} \approx 77.2$; b. $s \approx 10.4$; c. 75%; d. 0.25

Appendix I: Protractor

You can use a protractor to measure angles. If you do not have a protractor, you can make one with paper. Trace this protractor with a pen onto another piece of paper. Then, cut out the semi-circle using scissors.



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